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SYSTEM RELIABILITY UNDER MULTIPLE HAZARDS

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Structural reliability under multiple hazards is investigated. In the first part, the reliability of structural systems is studied with emphasis on the consideration of load sequence, load path and progressive failure over time. Currently available methods are critically examined and new methods of analysis are proposed for redundant systems based on an occurrence and coincidence consideration of loads and an imbedded Markov chain representation of damage state. Extensive Monte-Carlo simulations are carried out to verify the analytical methods. In the second part, several first and second order asymptotic methods for nonlinear load combination are examined and compared in terms of the analytical and numerical efforts required and the accuracy of each method. Numerical examples are carried out on the reliability of a structure with direction sensitive resistance under the action of vector wind force to underscore the advantages and disadvantages of each method in practical application.

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PART I--STRUCTURAL SYSTEM RELIABILITY UNDER TIME VARYING LOADS

1. INTRODUCTION

1.1 General

Great progress has been made in recent years in the analysis of reliability of structures under random loads. The description of structural performance is more clearly defined in terms of the limit state functions. The first order method in conjunction with efficient algorithms presents a consistent and practical way of evaluating structural reliability in terms of the limit state being reached. As a result, recent research interest has been more toward structural systems of considerable complexity. Primary considerations have been on treatment of structural configuration (being in series, or parallel or combination thereof); material properties (being brittle or ductile or of limited ductility); and evaluation of system reliability in terms of those of the members. Algorithms have also been developed such that the system reliability can be calculated with good accuracy and at a reasonable cost (2,6,7,8,12). However, thus far in most of the studies, the loadings have been idealized as time invariant, i.e., as random variables. In other words, the reliability so obtained corresponds to that under one load application. As most loads fluctuate in time, they may be "on" or "off" and may act individually or in combination with other loads. The reliability problem may be significantly different from under time invariant loads. For example, sequence and path of load application may be important, i.e., application of loadings in different order in time may produce different effects. The failure (collapse) of a system may be a culmination of progressive failures of members over a long period of time than a sudden failure of all members at one time. Such considerations dictate that the time domain fluctuation of loadings and their interaction with the system be included in the reliability analysis. Modeling of time varying loads and their individual and combined effect on structures as random processes has received much attention in recent years. Efficient models which are capable of representing both large and small scale fluctuation of loads have been developed (5,9,13,16). However, applications to reliability analysis have been quite isolated and limited to member reliability of linear structures, or simple ductile frames. The purpose of this study is to present a more systematic treatment of system reliability under time varying loads. Some of the currently available methods are critically examined, regarding their theoretical bases, computational effort required, accuracy that can be achieved and adequacy in representing real physical systems. New methods of analysis are also proposed based on a load coincidence consideration of the loadings and a Markov Chain treatment of the damage state of redundant systems. Extensive Monte-Carlo simulations are carried out to provide common background against which various approximate methods of analysis are compared.

1.2 Problem Definitions and Assumptions

The reliability problem to be investigated is the probability that a structural system, consisting of members (components), reaches a limit state, either collapse or a given state of damage, over a given period of time under the action of one or more time varying loads. It is well-known that depending on the scale of the fluctuation of the loadings, dynamic amplification of the structural response may become significant. Treatment of this problem requires the statistics of the dynamic response of the structure. It is also known as the dynamic load combination problem. For simple limit states such as those in terms of displacement of a linear structure, random vibration-based procedures have been presented (9,16). For inelastic structures, proper modeling of the structural hysteresis and deterioration as well as definition of damage and failure in terms of statistics of structural response are required. The recent developments of a random vibration method which includes these considerations but for structures under a single excitation has been summarized in Wen (15). The emphasis of this study is on the system aspect of the problem, i.e., effect of loading sequence, progressive failure over time, etc., therefore as a first step toward a better understanding of structural system reliability under multiple time varying loads, dynamic effect will not be considered. In other words the loadings are assumed to have large scale of fluctuation that the structures will respond statically or quasi-statically.

The loadings with macro-scale variability are modeled by pulse processes S(t) in which the load occurrence time, duration and intensity are treated as random variables (Fig. 1). The load occurrence time is assumed to follow a Poisson process. If independence is also assumed between the load intensity and duration in each occurrence as well as from occurrence to occurrence, a pulse process can be specified by a mean occurrence rate ν, a mean duration μ_d and an intensity random variable X with a density function $f_X(x)$. $\nu\mu_d$ varies from 0 to 1, representing the proportion of the time that the load is "on." Most transient loads have a $\nu\mu_d$ value <<1. The two extreme cases ($\nu\mu_d$ = 0 or 1) correspond to a Poisson spike process and a Poissson square wave process, respectively. The density function of the arbitrary-point-in-time value of the pulse process, S_i, has a mixed distribution with a discrete mass equal to $(1 - \nu_i \mu_d)$ at $S_i = 0$ (Fig. 1). It is a simple and flexible model which captures the main features of static loadings. Details and applications of this model can be found for example in Refs. 5 and 13.

In study of system reliability over a long period of time, whether a partially damaged structure is repaired or not before the next load application has important implications in the analysis. For example, reliability will depend on the repair time required and change in loading because of repair, etc. The same is true with the possible change with time in the structural resistance due to repair. In this study, to keep the analysis tractable, no repair is assumed; also the member resistance is assumed to be time invariant, i.e., until the limit state is reached. However, the no repair assumption will make the system resistance deteriorating with time because of possible progressive loss of members.

If not specifically mentioned, the loads that act on the structure are assumed to be statistically independent of one another, i.e., in terms of occurrence time, duration and intensity. The independence assumptions, however, can be relaxed. Models for dependent loads and the effects of the dependencies on combined load probability have been investigated in Wen (14). The effect of load dependencies on system reliability can be treated in a similar manner and will be a separate subject for future study.

2. TIME VARIANT RELIABILITY OF SERIES SYSTEMS

Herein, a series system means that the failure (collapse) of the system can be represented as events in series (union).

2.1 Weakest Link System

The simplest kind of series system is that of the weakest link type. (See Fig. 2.) Obviously under the action of time varying loads, the system reliability at a given time $t = t_o$ is that the weakest member has a resistance greater than the maximum combined load up to $t = t_o$. The reliability analysis therefore reduces to that of finding the probability distribution of the minimum among the member resistances and the maximum value of the combined load

$$L(t_o) = \int_0^{\infty} f_{R_{min}}(s)\, F_{S_{max}}(s,t_o)\, ds \tag{1}$$

in which,

$L(t)$ = reliability function;

$R_{min} = \min(R_1, R_2,\ldots,R_n)$, R_i = member resistance;

f = probability density function;

$S_{max} = \mathrm{Max}\,[S_1(t)+S_2(t)+\ldots S_m(t)]$ in $(0,t)$, $S_i(t)$ = load; and

F = probability distribution function.

$f_{R_{min}}$ can be derived from the joint density function of R_i. To determine $F_{S_{max}}$, the load coincidence method (9) may be used,

$$F_{S_{max}}(s,t_o) \approx \exp[-\{\sum_{i=1}^{m} \lambda_i F^*_{X_i}(s) + \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} \lambda_{ij} F^*_{X_{ij}}(s)$$

$$+ \sum_{i=1}^{m-2} \sum_{j=i+1}^{m-1} \sum_{k=j+1}^{m} \lambda_{ijk} F^*_{X_{ijk}}(s) + \ldots\} t_o] \qquad (2)$$

in which,

$$\lambda_i = \nu_i \prod_{j \neq i} (1 - \nu_j \mu_{d_j})$$

$$\lambda_{ij} = \nu_i \nu_j (\mu_{d_i} + \mu_{d_j}) \prod_{k \neq i \neq j} (1 - \nu_k \mu_{d_k})$$

$$\lambda_{ijk} = \nu_i \nu_j \nu_k (\mu_{d_i}\mu_{d_j} + \mu_{d_i}\mu_{d_k} + \mu_{d_j}\mu_{d_k}) \prod_{\ell \neq i \neq j \neq k} (1 - \nu_\ell \mu_{d_\ell})$$

are the occurrence rate of $S_i(t)$ only, joint occurrence rate of $S_i(t)$ and $S_j(t)$ only; and joint occurrence rate of $S_i(t)$, $S_j(t)$ and $S_h(t)$ only. $F^*_{X_i}(s) = 1 - F_{X_i}(s)$, probability of threshold level s being exceeded given the occurrence of load $S_i(t)$ only. $F^*_{X_{ij}}(s) = 1 - F_{X_{ij}}(s)$, $X_{ij} = X_i + X_j$, probability of threshold s being exceeded given the joint occurrence of $S_i(t)$ and $S_j(t)$. $F^*_{X_{ijk}}(s)$ is similarly defined.

2.2 Ductile Frames

The plastic collapse of a ductile frame can be treated as the union of all the possible failure modes (mechanism), therefore as a series system in a wide sense. Based on a first order plastic collapse (limit state) analysis, the modes can be identified and described in terms of a linear performance function of the load and resistance variables. However, since the loads are time varying, the reliability problem can be treated as no crossing of the vector load process out of the safe domain which is a function of the structural resistance against plastic collapse. Since the member resistance is assumed to be time invariant, for given (known) resistances of the members $\underset{\sim}{R} = \underset{\sim}{r}_o$ the failure domain for the system including all mechanisms is the union of all failure modes.

$$F = \bigcup_{i=1}^{\ell} F_i \tag{3}$$

in which,

$F_i = (\sum_{j=1}^{m} C_{ji} S_j > D_i)$, the i-th failure mode, ℓ = number of failure modes;

C_{ji} = coefficients determined from plastic collapse analysis;

S_j = intensity of j-th load; and

D_i = structural capacity against ith mechanisms, function of $\underset{\sim}{r}_o$.

The reliability of the system at $t = t_o$ is therefore the probability that there is no crossing of the vector process $\underset{\sim}{S}(t) = [S_1(t), S_2(t), \ldots, S_m(t)]$ into the failure domain in the $\underset{\sim}{S}$ space described by Eq. 3 over the period $(0, t_o)$. Note that no gravity effect is considered and it is assumed that member resistance does not deteriorate from repeated load applications, therefore even without repair, damage due to previous load applications such as formation of plastic hinges without collapse would not alter the failure domain. The problem as formulated above is that of the first passage probability. The solution is generally difficult. Approximate methods can be used if the statistics of outcrossing rate can be determined. For example, based on a Poisson outcrossing assumption, a good estimate of the reliability of the system is

$$L(t) \approx \exp\left[- \nu(\underset{\sim}{r}_o)\ t_o\right] \tag{4}$$

in which $\nu(\underset{\sim}{r}_o)$ = the mean outcrossing rate given the resistance $\underset{\sim}{R} = \underset{\sim}{r}_o$.

In this investigation, since the load processes are assumed to be independent Poisson pulse processes, the outcrossings would also follow a Poisson law, therefore solution given in Eq. 4 is exact. Other approximate methods, such as those based on a bounding technique can also be used (3,11). The crossing analysis of a vector process into a union of failure domains as given in Eq. 3, however, is not elementary. Rackwitz (10) proposed an approximate method for loads of the square wave type, (i.e., $\nu_i \mu_{d_i}$ = 1.0 for all i).

$$\nu = \sum_{i=1}^{m} \nu_i \left[P(\underset{\sim}{X}_+^i \ \varepsilon\ F) - P\left(\{\underset{\sim}{X}_+^i \ \varepsilon\ F\} \cap \{\underset{\sim}{X} \ \varepsilon\ F\}\right)\right] \tag{5}$$

in which,

$\underset{\sim}{X}$ = vector of load intensity random variables; and

$\underset{\sim}{X}_+^i$ = vector of load intensity random variables given there is a change in the i-th load component.

The probability in the bracket is that of the process within the domain F before the change in X^i and out of F after the change in X^i, i.e., an outcrossing. Therefore, the product gives the outcrossing rate. The calculation of the probability terms involved, however, requires evaluation of multi-variate normal distribution of order of 2ℓ, ℓ = number of failure modes considered. An algorithm has been developed for the evaluation of this distribution approximately. Details can be found in Ref. 4. For the more general pulse processes in this study, this method is generalized by using a mixed density function for the load intensity. Details and results are given in the Appendix.

When the resistances are unknown and modeled by random variables, Eq. 4 gives the conditional reliability for $\underset{\sim}{R} = \underset{\sim}{r}_o$. The unconditional reliability is therefore

$$L(t_o) = \int_{\underset{\sim}{R}} \exp[-\nu(\underset{\sim}{r})\ t_o]\ f_{\underset{\sim}{R}}(\underset{\sim}{r})\ d\underset{\sim}{r} \tag{6}$$

in which $f_{\underset{\sim}{R}}$ = joint density function of the resistance variables. Note that a n-fold integration is required where n = number of resistance random variables considered. Numerical integration becomes impractical for $n > 3$. Approximate methods are therefore necessary under these circumstances.

2.2.1 <u>Approximate Outcrossing Rate Method</u>

If one includes the resistance variability by modeling them as random variables in the outcrossing rate evaluation (Eq. 5), one in essence obtains the ensemble average (over the resistances) of the mean outcrossing rate. This method has been suggested in Ref. 10 in an upper bound estimate of the failure probability

$$p_f \leqq \nu \, t_o + p_f(0) \tag{7}$$

in which ν is the ensemble average of the mean outcrossing rate and $p_f(0)$ is the probability of collapse at $t = 0$. It is obvious that in connection with this ensemble outcrossing rate the reliability based on the Poisson outcrossing assumption

$$L(t_o) = \exp[- \nu \, t_o] \tag{8}$$

is strictly no longer valid. However, Eq. 8 generally gives a better and still conservative approximation. Accuracy of Eq. 8 will be examined.

2.2.2 Load Coincidence Method

Taking advantage of existing methods of time invariant system reliability analysis and the load coincidence method for load combination Wen (13), proposed the following approximate method

$$L(t_o) \approx \exp(- \nu \, t_o)$$

in which

$$\nu = \sum_{j=1}^{m} \lambda_j \, P_j + \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} \lambda_{ij} \, P_{ij} + \sum_{i=1}^{m-2} \sum_{j=i+1}^{m-1} \sum_{k=j+1}^{m} \lambda_{ijk} \, P_{ijk} + \dots \tag{9}$$

λ_j, λ_{ij} and λ_{ijk} are defined as in Eq. 2. P_j, P_{ij} and P_{ijk} are correspondingly the conditional probabilities of structural failure (collapse) given the occurrence of $S_i(t)$ only, joint occurrence of $S_i(t)$ and $S_j(t)$ only and joint occurrence of $S_i(t)$, $S_j(t)$ and $S_k(t)$ only. Given the occurrence of the loads, these conditional probabilities can be evaluated based on state-of-the-art method of analysis for time invariant system reliability. This relatively simple formulation can be applied to rather complex systems without difficulty since methodologies for time invariant reliability analysis are well developed. Implied in the above approximation, however, is the assumption that failures (collapse) are independent events (Poisson assumption). It is strictly not valid since the resistance variables remain the same rather than change independently from one load application to the next as would be required in a Poisson failure process. Therefore the approximation is similar to that in using the ensemble average of the mean outcrossing rate where the dependence through the structural resistance from occurrence to occurrence has been neglected. It tends to overestimate the failure probability.

2.3 Numerical Examples

To illustrate the various methods of analysis and to compare the accuracies of these methods, numerical examples on the plastic collapse of a single frame under vertical and horizontal time variant loads are carried out. The geometry and the member properties of the frame are shown in Fig. 3 and Table 1. The loadings are sparse Poisson pulse process representing transient loads. The loading parameters are also given in Table 1. The possible mechanisms are shown in Table 2. The ensemble failure rates then the reliability as a function of time based on different methods are compared. Exact solutions are obtained as benchmarks based on numerical evaluations of Eq. 5 and Eq. 6 and

$$\nu = \int_{\underset{\sim}{r}} \nu(\underset{\sim}{r})\ f_{\underset{\sim}{R}}(\underset{\sim}{r})\ dr \tag{10}$$

where $\nu(\underset{\sim}{r})$ = conditional outcrossing rate for $\underset{\sim}{R} = \underset{\sim}{r}$ (Eq. 5). Some details of the outcrossing rate evaluation can be found in Ref. 13. Only three failure modes (Numbers 1, 2 and 5) were used in the approximate outcrossing rate method by Rackwitz; even so calculation of 6th order multi-variate normal distribution functions are needed. (See Appendix.) In the load-coincidence approximation, the conditional probabilities of failure are obtained using the point estimate based on the probability network method by Ang and Ma (1). This method is used because of its simplicity and good accuracy; only hand calculations are required for this problem. Other methods may be used if more accuracy is needed (2). Some details of calculation are given in Table 3. This method also allows a direct evaluation of the contribution due to load coincidence. The ensemble outcrossing rates are compared in Table 3. It is seen that both approximations give satisfactory results. Computationally the load coincidence method is least demanding. It can be seen from the table that only 7.4% of the total failure can be attributed to the simultaneous action of the two loads because of their transient nature (mean duration $\mu_d = 10^{-2}$ yr., or 3.6 days). If the mean load duration increases to 10^{-1}

yr. (36 days); the failure rate increase to $\nu = 4.99 \times .00128 + 0.190 \times 0.0135 + 0.2 \times 0.036 = 0.0161$ of which the load coincidence accounts for 45%.

The reliability function using the ensemble failure rate (Eq. 8) is compared with the exact solution (Eq. 6) in Fig. 4. It is seen that Eq. 8 consistently underestimates the system reliability, the error can be significant as time increases. As an independent check of the results of the analytical methods, Monte Carlo simulations were also carried out. In the simulation study, the resistance variables are kept the same as those given in Table 1. Three additional loading conditions, however, are added. A sample size of 800 is used. The probabilities of failure (collapse) within the first year based on different methods are compared in Table 4. The simulation results agree closely with the exact solution. As expected the approximate (load coincidence or outcrossing rate) method overestimates the failure probability, the error is comparable to that shown in Fig. 4, i.e., it is small when the failure probability is small, but tends to increase with the failure probability. This is largely due to the dependence through the resistance which is not considered in the approximate methods.

3. TIME VARIANT RELIABILITY OF PARALLEL SYSTEMS

A typical parallel system under multiple time varying load is shown in Fig. 5. An almost trivial case is when the members have resistance of ideal ductile (elasto-plastic) behavior. The system resistance is therefore the sum of those of the members and the reliability at $t = t_o$ is

$$L(t_o) = P\left[\sum_{i=1}^{n} R_i > \sum_{j=1}^{m} S_i(t) \text{ in } (0,t_o)\right] = P[R_S > S_{max}] \tag{11}$$

in which

R_i = resistance of i-th member;

$R_S = \sum_{i=1}^{n} R_i$ = system resistance; and

$S_{max} = \text{Max}[S_1(t) + S_2(t) + \ldots S_n(t), \text{ in } (0,t_o)]$

As in Eq. 1, the system reliability is

$$L(t_o) = \int_0^{\infty} f_{R_S}(s)\, F_{S_{max}}(s,t_o)\, ds \tag{12}$$

The problem is considerably more complex when the member resistance is brittle; primarily due to the fact that the sequence of member failure becomes important and progressive failure of member over time needs to be considered. It can be seen that depending on the loading time domain behavior, member resistance, and load distribution among the members, the interaction of the system with the loadings can be extremely complicated. It is difficult to describe the system reliability in general terms, therefore an analytically more tractable procedure of from simple to complex systems is followed.

3.1 System with Equal Load Distribution

Referring again to Fig. 5, if there are additional constraints in the system (not shown), such that the load is equally distributed among the members of the system, and if a member fails, the load will be distributed evenly among the rest of the members and so on. For this class of parallel systems, the reliability can be formulated as follows.

Consider a simple system of two members, with member resistance R_1 and R_2, respectively. For $R_1 = r_1$, and $R_2 = r_2$ and under the condition $r_2 > r_1$. Since the loads are equally distributed, member number 1 always fails first. Within a time period t_o, the system may fail (collapse) according to the sequence shown in either Fig. 6-a or Fig. 6-b. In the former case $r_2 > 2r_1$, the first member fails first, indicated by an "x," and followed by the 2nd member, indicated by a "0," at a later time, whereas in the latter case $r_2 < 2r_1$, as soon as the first member fails, the second member follows instantaneously. In either case, the event of system collapse is described by

$$E = [S_{max} > 2r_1 \cap S_{max} > r_2] \tag{13}$$

where

$$S_{max} = \max[S_1(t) + S_2(t) + \dots S_m(t) \text{ in } (0,t_o)]$$

For $r_2 < r_1$, the sequence of member failure is reversed, therefore the failure event is

$$E = [S_{max} > 2r_2 \cap S_{max} > r_1] \tag{14}$$

The probability of system collapse is therefore the sum of the two mutually exclusive events

$$P_f = \int_0^\infty [\int_0^{S/2} \int_{r_1}^{S} f_{R_1R_2}(r_1,r_2)\ dr_2 dr_1 + \int_0^{S/2} \int_{r_2}^{S} f_{R_1R_2}(r_1,r_2)\ dr_1 dr_2]$$

$$f_{S_{max}}(s)\ ds \qquad (15)$$

$f_{S_{max}}(s)$ can be obtained from the load coincidence method (Eq. 2). One would recognize that the formulation is identical to that of time invariant problems if the load random variable is the maximum combined value. Therefore, existing methods of time invariant system reliability analysis (1,2,6,7,8) can be directly applied to this class of problems. Also, through the formulation of the load coincidence method, the dependencies within and between loads can be also included into the reliability analysis (14).

This problem can be alternatively solved from an outcrossing formulation. For given resistances, the failure (outcrossing) occurs when $S(t) = \sum_{i=1}^{m} S_i(t)$ upcrosses max $(2r_1,r_2)$ for $r_2 > r_1$, or max $(2r_2,r_1)$ for $r_1 > r_2$. The upcrossing rate conditional on the resistance can be obtained and the reliability can then be evaluated according to Eq. 4. As can be seen, however, the computation required becomes unmanageable as the number of members increases. The approximate outcrossing method (10) can also be used, by which the ensemble average of the mean outcrossing rate of a vector process into a (system) failure domain which is an intersection of (member) failure domains can be obtained. This approach will be discussed in greater detail in the following.

3.2 System with Unequal Load Distribution

In a more general and perhaps more realistic parallel system, the distribution of the loads among the members may not be uniform, i.e., members carry unequal shares of the loadings that are acting on the system. The sequence of member failure is still more complicated since the weaker member may not fail first. Furthermore, when using an outcrossing formulation, one may not be able to find a time invariant cut set representing the system failure domain. That is, the failure domain becomes loading path dependent, thus is itself time variant. To illustrate this point, consider a simple case of two members with resistance $R_1 = 1$ and $R_2 = 2$. The system is under the action of two time varying loads $S_1(t)$ and $S_2(t)$. The resultant force in each member is given by

$$F_i(t) = C_i \; S_1(t) + D_i \; S_2(t) \qquad\qquad i = 1,2 \qquad\qquad (16)$$

in which

$$C_1 + C_2 = 1 \qquad , \qquad D_1 + D_2 = 1$$

If one member fails, the other member carries the whole load. Assume the load distribution is such that $C_1 = 0.3$, $D_1 = 0.7$. The failure surfaces (lines) for each member in the S_1, S_2 plane are shown in Fig. 7. The solid lines indicate the failure sequence of member number 1 then number 2 and the reversed sequence is shown by dashed lines. Indicating the loss of a member by an "x" and the collapse of the system by an "0," it is seen that depending on the load path, the system collapse may be instantaneous, or the system may be "damaged" first and there is considerable time elapsed before collapse occurs. Also, perhaps more importantly, the failure domain is load path dependent. Disregarding this dependence, the outcrossing result will be in error. For example, if the system failure is defined as the vector load process outcrosses into the cut set (hatched area in Fig. 7),

$$\left(.3\, S_1 + .7\, S_2 > 1 \cap S_1 + S_2 > 2\right) \cup \left(.7\, S_1 + .3\, S_2 > 2 \cap S_1 + S_2 > 1\right) \tag{17}$$

the failure rate will be underestimated since for certain load paths the system failure domain is

$$S_1 + S_2 > 2 \tag{18}$$

It can be easily seen that under the assumption of uniform load distribution among the members, the failure surfaces are parallel to one another, the failure domain is independent of the load path. Therefore results derived from the case of equal load distribution cannot be generalized to the case with unequal load distribution. This is also true in the formulation based on the maximum combined value. For example, to generalize the result given in Eq. 13, for the sequence of failure of member number 1 then 2, the event of system collapse would be

$$E = \{\max[.3\, S_1(t) + .7\, S_2(t)] > 1\} \cap \{\max[.7\, S_1(t) + .3\, S_2(t)]$$

$$< 2\} \cap \{\max[S_1(t) + S_2(t)] > 2\} \qquad (19)$$

However, it may so happen that for some load paths (Fig. 8) the last event occurs before the first and the system still stands. The failure probability therefore would be overestimated by Eq. 19.

3.2.1 Imbedded Markov Chain Model

To include the loading path dependence, the sequence of failure of members and the redistribution of loads among the members from application to application needs to be considered. A formulation based on a Markov Chain description of the states of various stages of damage of the system including collapse is presented in the following.

Consider first the case that the member resistances are known (deterministic). The state of the system in terms of the failure or survival of the members may change only at the occurrence (application) of loads, i.e., either individual load or loads in combination. Since the load intensities are assumed to be independent from occurrence to occurrence within each load and among different loads, the state of the structure at any load application depends only on that at the previous load application and not the ones before; that is, if the dependence of load through the consecutive occurrences of one load during the occurrence of another load is neglected. For most transient loads, $\nu\mu_d << 1$, this load dependence is small. Also, the dependence generally tends to cause a lower combined effect (14). Therefore, neglecting this dependence tends to give slightly conservative estimates of the failure probability and the state of the structure changes can be modeled at least approximately as a Markov Chain from load application to application. Since the loading occurrences are modeled by independent Poisson processes, the occurrence time for the structure state changes is also a Poisson process in which the Markov chain is imbedded. Given the load occurrence (or coincidence), the transition probability of one state to another can be evaluated based on currently available time invariant system reliability analysis and a load coincidence analysis. The probabilities of the various states and the reliability of the system as functions of time can be then evaluated using available Markov process results.

Consider a parallel system with n members, the state of the system is described in terms of all possible configurations (damaged or intact) that can carry the loads. The case of n = 3 is shown in Fig. 9 where failed members are indicated by dashed lines. At each load application, the system may or may not change into a less stable state (with more member failures). These transition probabilities are functions of the load intensity random variables only since resistances are known. By considering load coincidence, the transition probability from state I to J given the occurrence of loads is

$$P(J|I) = \frac{1}{\lambda}\Big[\sum_{i=1}^{m} \lambda_i P_{JI}^{i} + \sum_{i=1}^{m-1}\sum_{j=i+1}^{m} \lambda_{ij} P_{JI}^{ij} + \sum_{i=1}^{m-2}\sum_{j=i+1}^{m-1}\sum_{k=j+1}^{m} \lambda_{ijk} P_{JI}^{ijk} + \ldots\Big] \qquad (20)$$

in which λ_i, λ_{ij}, λ_{ijk}, etc. are defined in Eq. 2

$$\lambda = \sum_{i=1}^{m} \lambda_i + \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} \lambda_{ij} + \sum_{i=1}^{m-2} \sum_{j=i+1}^{m-1} \sum_{k=j+1}^{m} \lambda_{ijk} + \cdots$$

the overall occurrence rate of loads including joint occurrences;

P^i_{JI} = probability of transition from state I to J given only load $S_i(t)$ is "on";

P^{ij}_{JI} = probability of transition from state I to J given only load $S_i(t)$ and $S_j(t)$ are "on";

etc.

In Eq. 20 the ratio of the occurrence rates give the probabilities of different combinations of loads based on a frequency interpretation. For example, λ_{ij}/λ is the probability of coincidence of loads $S_i(t)$ and $S_j(t)$ given the occurrence of loads. P_{JI} terms can be evaluated using currently available methods.

Since no repair is allowed, the damage process is irreversible. The transition matrix is a lower triangular one. Also, many of the lower triangular elements are zero. For example, in the 7 x 7 transition matrix for the 3-member system,

$$P(3|2) = P(4|2) = P(4|3) = P(5|4) = P(6|3) = P(6|5) = P(7|2)$$
$$= P(7|5) = P(7|6) = 0$$

Let the transition matrix be [A] and the probability of the initial state of the structure be described by the column matrix {P(0)}, the probability of the structural state at time $t = t_o$ is therefore

$$\{P(t_o)\} = \sum_{k=0}^{\infty} [A]^{k+1}\{P(0)\} \frac{(\lambda t)^k}{k!} e^{-\lambda t_o}$$

$$= [A][Q][\ulcorner e^{-\lambda t_o(1-u_i)} \lrcorner][Q]^{-1}\{P(0)\} \tag{21}$$

in which [Q] = the eigen vector of [A], $[\ulcorner\ \lrcorner]$ diagonal matrix with element $e^{-\lambda t_o(1-u_i)}$; and u_i = the i-th eigenvalue of [A], i.e., the i-th diagonal term of [A].

The reliability (no collapse) of the system is therefore the sum of the probabilities of all the (noncollapse) states in the column matrix {P(t)}

$$L(t_o) = [1]^t[A]\{P(t_o)\} \tag{22}$$

in which $[1]^t$ = transpose of a unit column matrix. The hazard function, which is often of interest, is

$$h(t_o) = \frac{[1]^t[A][Q]\ [\ulcorner \lambda(1-u_i)\ e^{-\lambda t_o(1-u_i)} \lrcorner][Q]^{-1}\{P(0)\}}{L(t_o)} \tag{23}$$

For example, as the structure is weakened by progressive losses of members, the hazard function is expected to be an increasing function of time. In this regard, it is also obvious that results based on an outcrossing formulation assuming a time invariant failure domain will give a hazard function (failure rate) that is always time invariant.

When the member resistances are random variables, the reliability can be evaluated by integrating over the resistance variables

$$L(t) = \int_{\underset{\sim}{X}} L(t,\underset{\sim}{x})\ f_{\underset{\sim}{X}}(\underset{\sim}{x})\ d\underset{\sim}{x} \tag{24}$$

in which $\underset{\sim}{X}$ = vector of random variables representing the resistance. Again, as the dimension of $\underset{\sim}{X}$ increases, the computation can easily get out of hand. Approximations may be necessary. One simple approximation based on Taylor series expansion gives the reliability as function of the moments of $\underset{\sim}{X}$ as follows

$$L(t) \approx L\left(t,\mu_{\underset{\sim}{X}}\right) + \frac{1}{2!} \sum\sum \left.\frac{\partial^2 L(t,\underset{\sim}{x})}{\partial x_i \partial x_j}\right|_{\mu_{\underset{\sim}{X}}} \rho_{x_i x_j} \sigma_{x_i} \sigma_{x_j}$$

$$+ \text{ higher order terms} \tag{25}$$

in which the partial derivatives may be evaluated numerically using a finite difference scheme.

An alternative approximation is to assume that the resistance also vary independently from load application to application. Therefore, the resistance variability can be included in the transition probability, i.e., instead of constant resistances, random variables are used in evaluating terms P_{JI} in Eq. 20.

In the outcrossing rate approach to the problem, if the load path dependence is disregarded and the resistance variability is included through an use of the ensemble outcrossing rate, then the failure probability can be evaluated by Eq. 7 or Eq. 8. This method has been suggested by Rackwitz (10) for systems under Poisson square wave type of loads. For pulse type of loads, the method can be extended as follows.

3.2.2 Approximate Outcrossing Rate Method

If the failure domain can be expressed as a cut set

$$F = \bigcup_{k=1}^{\ell} F_k = \bigcup_{k=1}^{\ell} \left(\bigcap_{r=1}^{n} f_{kr} \right) \tag{26}$$

in which

f_{kr} = failure domain for r-th member in failure sequence (mode) k;

n = number of members; and

ℓ = number of failure sequence (mode).

For Poisson square waves, the outcrossing rate is given by

$$\nu = \sum_{i=1}^{\ell} \nu_i \, P[(\underset{\sim}{x}_+^i \in F) \cap (\underset{\sim}{x} \in \bar{F})] \tag{27}$$

Notations are the same as in Eq. 5, however, the failure domain is more involved. After some approximations, one obtain from Eq. 26 and Eq. 27

$$\nu \approx \sum_{k=1}^{\ell} \sum_{i=1}^{m} \nu_i \Big[P\Big(\underset{\sim}{x}_+^i \in \bigcap_{r=1}^{n} f_{kr}\Big) - P\Big\{ \bigcup_{j=1}^{\ell} \Big(\Big(\underset{\sim}{x}_+^i \in \bigcap_{r=1}^{n} f_{kr}\Big) \cap \Big(\underset{\sim}{x} \in \bigcap_{r=1}^{n} f_{jr}\Big)\Big)\Big\}\Big] \tag{28}$$

The probabilities required in Eq. 28 can be evaluated using the methods given in Ref. 10 in which a bounding technique is also required. Extension to load process modeled by pulse processes can be carried out as indicated in the Appendix. It is pointed out that neglecting the dependence of failure domain on load path, one generally underestimates the failure rate; on the other hand, using the ensemble outcrossing to include the load resistance one generally overestimates the failure probability. The overall effect of this approximate outcrossing method is therefore not clear, it may over or underestimate the failure probability, even assuming Eq. 28 can be evaluated exactly.

3.2.3 Load Coincidence Method

If the deterioration of the system resistance with time and dependence of loading path are neglected as in the approximate outcrossing formulation, the failure (collapse) rate can also be evaluated based on a load coincidence consideration as given in Eq. 9. The required conditional probabilities of failure are correspondingly the system collapse probabilities given the occurrence or coincidence of loads. P_j, P_{ij}, and P_{ijk} therefore can be evaluated directly based on a method for time invariant problems. Although the theoretical basis for this approach is not particularly appealing, the computational aspect of this method is quite attractive, especially compared with the outcrossing formulation. Its accuracy will be examined in the following; since the approximations involved are comparable to those in the outcrossing method, comparable accuracy is expected.

3.3 Numerical Examples

It is clear that except for the case where loads are distributed uniformly among the members, approximations are required in all analytical methods for parallel brittle systems. The accuracies of each method and the interpretation of the results are therefore important considerations when such methods are used. Consider first a simple two-member parallel system under the combined action of two loads. The load distribution is uniform and the member resistances are independent and have identical distributions, i.e., $f_{R_1}(r_1) = f_{R_2}(r_2) = f_R(r)$. Under these conditions, Eq. 15 reduces to

$$P_f = 2 \int_0^\infty \{F_R(s)\ F_R(s/2) - \tfrac{1}{2}[F_R(s/2)]^2\}\ f_{S_{max}}(s)\ ds \qquad (29)$$

in which

$$S_{max} = \max[S_1(t) + S_2(t)] \text{ in } (0,t)$$

$f_{S_{max}}(s)$ can be derived from $F_{S_{max}}(s)$ given in Eq. 2. The solution is "exact" to the extent that the maximum combined load distribution is given by Eq. 2. The structural and loading system parameters are given in Table 5. All random variables are assumed to have normal distribution. The reliability as a function of time is shown in Fig. 10. The hazard function is also evaluated numerically and shown in the same figure. As has been mentioned in the foregoing that for this particular system with member carrying equal loads, if the resistances are deterministic (known), the system failure domain is time invariant, hence so is the failure (outcrossing) rate (or hazard function). However, for random resistance, as can be seen the hazard function becomes a decreasing function of time. This rather unexpected behavior of the hazard function for random resistance can be also analytically derived and verified for a simple single member system under a single Poisson pulse loading.

The same problem is also solved using the Markov process model in which the random resistance is included in the transition probability evaluation. The results are also given in Fig. 10. As expected, largely due to this approximation of the resistance variabilities, the failure probability is overestimated. Also, the hazard function remains almost at a constant level. The approximate outcrossing rate method based on an extension of Eq. 28 is also used to solve the problem. This method gives a constant ensemble failure rate. The lower bound solution according to Eq. 7 is also shown in Fig. 10. In conjunction with a Poisson failure occurrence, this method overestimates the failure probability and the accuracy is comparable to that based on the Markov approach. Again, this is expected since the dependence of the failure event through common resistance of the structure is neglected in both methods. Note that if the resistance variability is included through Eq. 24, both methods would be expected to yield solutions very close to the exact solution.

The structure investigated next is a two-member system but with members carrying unequal loads. Consider first the case that the member resistances are deterministic. The system parameters are given in Table 6. The various states that the system can carry the load (stable configuration) are shown in Table 7. The failure domains corresponding to each system state transition in the Markov approach are also given in Table 7. The transition probability matrix according to Eq. 20 and Table 7 for $R_1 = 1$, $R_2 = 2$ is

$$A = \begin{bmatrix} .94911 & 0 & 0 \\ .03139 & .95695 & 0 \\ 0 & 0 & .06811 \end{bmatrix}$$

The reliability as a function of time and the corresponding hazard function according to Eqs. 21, 22 and 23 are given in Fig. 11. Note that for this system, the progressive loss of members over time is possible and likely to occur, this is reflected by the increase of the hazard function with time. The outcrossing rate method would underestimate the failure rate because of its failure to include the dependence of the system failure domain on loading path into consideration. This error will be very serious if the parameters of the system are such that sequential failure over time is dominant. To illustrate this point, consider a limit case when $\mu_{d_1} = \mu_{d_2} => 0$, i.e., the loadings reduce to Poisson spike process under this condition. The Markov formulation becomes exact, since the possibility of load correlation is eliminated. The outcrossing rate of the vector process into the failure domain for this case is simplified to

$$\nu = \sum_{i=1}^{\ell} \lambda_i \; P[(\underset{\sim}{x}_+^i \; \varepsilon \; F)] \tag{30}$$

where

$$F = (\cdot 3\, S_1 + \cdot 7\, S_2 > 1 \cap S_1 + S_2 > 2) \cup (\cdot 7\, S_1 + \cdot 3\, S_2 > 2$$

$$\cap S_1 + S_2 > 1)$$

Therefore, the failure rate (hazard function) ν can be easily evaluated as

$$\nu = 2 \times P[S_1 > 2.857] + 2 \times P[S_2 > 2.0]$$

$$= 2 \times .57 \times 10^{-7} + 2 \times .33 \times 10^{-5}$$

$$= .67 \times 10^{-5}$$

Whereas in the Markov method, the transition probability matrix is easily evaluated as

$$A = \begin{bmatrix} .9675 & 0 & 0 \\ .0325 & .9757 & 0 \\ 0 & 0 & .070 \end{bmatrix}$$

From Eqs. 21 and 22 the reliability function can be obtained

$$L(t) = 4.033\, e^{-.0972t} - 3.033\, e^{-0.13t} \tag{31}$$

The reliability and hazard functions are also shown in Fig. 11. The outcrossing method so grossly underestimates the failure probability, that it is useless. For example, at t = 10 yrs., the outcrossing rate analysis would give an "upper bound" to failure probability of

$p_f \leqq .67 \times 10^{-4}$, whereas the actual failure probability is $p_f = .298$, and the failure rate at this time is 0.0593.

For $\mu_{d_i} \neq 0$, particularly when $\nu_i \mu_{d_i}$ becomes large for all loads, the coincidence of loads is frequent and the combined load intensity may be correlated from one coincidence to the next since the intensity of one load may not change during two consecutive load coincidences. This correlation has not been considered in the Markov formulation. To quantify the effect of this correlation, as well as to evaluate the accuracy of the outcrossing rate method as $\nu_i \mu_{d_i}$ becomes large. Monte Carlo simulations are carried out. The system parameters are given in Tables 8 and 9 where several combinations of load parameters are tried. The results are shown in Figs. 12 and 13. The range of $\nu\mu_d$ values varies from 0.002 ("sparse" pulse) to 1.0 (Poisson square wave, the load is "on" all the time). It is seen that the Markov approach gives excellent results for small values of $\nu\mu_d$ and may be quite conservative (overestimate the failure probability) for $\nu\mu_d \Rightarrow 1.0$. The outcrossing rate method, on the other hand underestimates the failure probability for small $\nu\mu_d$ values as expected, since the dependence of a failure damage on load path is neglected. This error becomes very serious for sparse pulse ($\nu\mu_d < 0.01$), therefore this method should not be used for such type of loadings unless the failure probability is very small. For large values of $\nu\mu_d$, however, the Poisson outcrossing assumption used in Eq. 8 neglects the strong correlation in outcrossings and tends to overestimate the failure probability. The overall effect is not clear, the result may or may not be on the conservative side.

The accuracy of the load coincidence method (Eq. 9) for parallel brittle systems is examined by comparing the results with those based on the approximate outcrossing rate method for a wide range of combinations of system parameters, since the approximations involved in these two methods are quite similar. The results are shown in Table 10. It is seen that for sparse pulse processes ($\nu\mu_d << 1$), they are almost indistinguishable; for large $\nu\mu_d$, the load coincidence method gives a higher estimate of the failure rate.

4. SUMMARY AND CONCLUSIONS

An investigation of the structural system reliability under time varying loads is carried out with emphasis on the consideration of load sequence, load path, and progressive failure over time. Currently available methods are critically examined and new methods of analysis are proposed for redundant systems based on an occurrence and coincidence consideration of loads and an imbedded Markov Chain representation of damage state. Extensive Monte Carlo simulations are carried out. Based on the simple systems examined, it is found that:

(1) The load coincidence method and the approximate outcrossing approach, although having dissimilar theoretical backgrounds, produce results with comparable accuracy; i.e., for both series systems and parallel brittle systems, however, with the former method being computationally much less demanding.

(2) For parallel brittle systems, if loads are equally shared by members, the problem can be reduced to a time invariant one provided the distribution of the maximum combined load over a given time period can be found. Current available methods for load combination can be used for such purpose.

(3) If the loads are unequally distributed among the members in a parallel brittle system, the system failure domain may be load path dependent and cannot be represented as a cut set of member failure domains. The outcrossing rate approach therefore inherently underestimates the failure (collapse) rate, and the error can be very serious when the loads are infrequent and brief.

(4) The imbedded Markov Chain model adequately treats the deterioration of the redundant system from load application to application and consistently gives good, conservative results.

(5) The failure (collapse) probability is always overestimated, sometimes severely, if the resistance variability is incorporated in using an ensemble failure rate. More accurate and computationally efficient methods need to be developed.

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TABLES

Table 1--System Parameters

	μ	σ	ν	μ_d
M_1	360^{k-ft}	54^{k-ft}	-	-
M_2	480^{k-ft}	72^{k-ft}	-	-
S_1	100^k	10^k	5/yr	10^{-2} yr
S_2	50^k	15^k	.2/yr	10^{-2} yr

Table 2--Possible Mechanism

1	$4\ M_1 + 2\ M_2 - 10\ S_1 - 15\ S_2$
2	$4\ M_1 - 15\ S_2$
3	$2\ M_1 + 4\ M_2 - 10\ S_1 - 15\ S_2$
4	$3\ M_1 + M_2 - 15\ S_2$
5	$4\ M_2 - 10\ S_1$
6	$2\ M_1 + 2\ M_2 - 15\ S_2$
7	$M_1 + 3\ M_2 - 10\ S_1$
8	$2\ M_1 + 2\ M_2 - 10\ S_1$

Table 3--Ensemble Mean Failure (Collapse) Rate ν

	ν	
Load Coincidence Method (Eq. 9)	.00967	$\lambda_1 = 4.99$ $\lambda_2 = 0.19$ $\lambda_{12} = 0.02$ $P_1 = 0.00128$ $P_2 = 0.0135$ $P_{12} = 0.036$
Exact Sol. (Eq. 10)	.0103	$\nu_1 = 0.00285$ $\nu_2 = .00748$ $\nu = \nu_1 + \nu_2$
Approximate Outcrossing Method (Eq. 5)	.00937	

Table 4--Failure Probabilities

Load Parameters	Approximate Method (Eq. 9)	Exact Solution (Eq. 6)	Monte Carlo Simulation (n = 800)
$\nu_1 = 5/yr$ $\nu_2 = .2/yr$ $\mu_{d_1} = \mu_{d_2} = .0/yr$	.0097	.0073	.0075
$\nu_1 = \nu_2 = 5$ $\mu_{d_1} = \mu_{d_2} = .01$	.084	.065	.060
$\nu_1 = \nu_2 = 20$ $\mu_{d_1} = \mu_{d_2} = .01$	.37	.21	.21
$\nu_1 = \nu_2 = 20$ $\mu_{d_1} = \mu_{d_2} = 0.025$	.51	.27	.28

Table 5--System Parameters

	μ	σ	ν	μ_d
R_1	100	15	–	–
R_2	100	15	–	–
$S_1(t)$	100	10	5/yr	.01 yr
$S_2(t)$	80	20	1/yr	.01 yr

Table 6--System Parameters

	μ	σ	ν	μ_d
R_1	1	0	–	–
R_2	2	0	–	–
$S_1(t)$	1.5	.3	2/yr	.01 yr
$S_2(t)$	1.2	.15	2/yr	.01 yr

Table 7--Failure Domain for System State Transition

1→1	$R_1 > .3\,S_1 + .7\,S_2 \cap R_2 > .7\,S_1 + .3\,S_2$
1→2	$R_1 < .3\,S_1 + .7\,S_2 \cap R_2 > .7\,S_1 + .3\,S_2 \cap R_2 > S_1 + S_2$
1→3	$R_1 > .3\,S_1 + .7\,S_2 \cap R_2 < .7\,S_1 + .3\,S_2 \cap R_1 > S_1 + S_2$ $= \phi$ for $R_1 = 1$ and $R_2 = 2$
2→2	$R_2 > S_1 + S_2$
2→3	ϕ
3→3	$R_1 > S_1 + S_2$

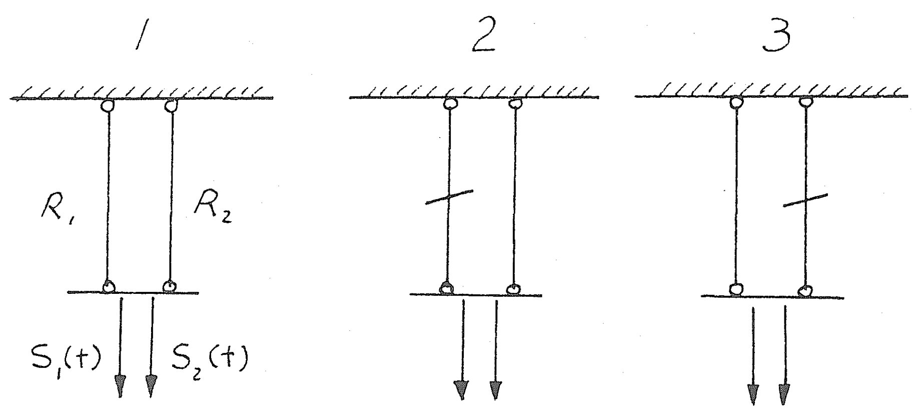

Table 8--System Parameters

	μ	σ	ν	μ_d
R_1	1.0	0	–	–
R_2	2.5	0	–	–
$S_1(t)$	2.0	.4	2	.001,0.01,0.1
$S_2(t)$	1.5	.3	5	.001,0.01,0.1

C_1 = .2 D_1 = .6

Table 9--System Parameters

	μ	σ	ν	μ_d
R_1	1.5	0	–	–
R_2	3.0	0	–	–
$S_1(t)$	1.5	.3	2	.01,.1,.5
$S_2(t)$	1.2	.15	2	.01,.1,.5

C_1 = .3 D_1 = .7

Table 10--Accuracy of Load Coincidence Method for Parallel Brittle Systems

ν System Failure (Collapse) Rate/Yr

μ_d	Approximate Outcrossing (Eq. 28)	Load Coincidence (Eq. 9)
.001 yr	.0078	.0078
.01 yr	.077	.078
.1 yr	.63	.77
System Parameters Given in Table 8		
.01 yr	.005	.005
.10 yr	.047	.050
.50 yr	.185	.250
System Parameters Given in Table 9		

FIGURES

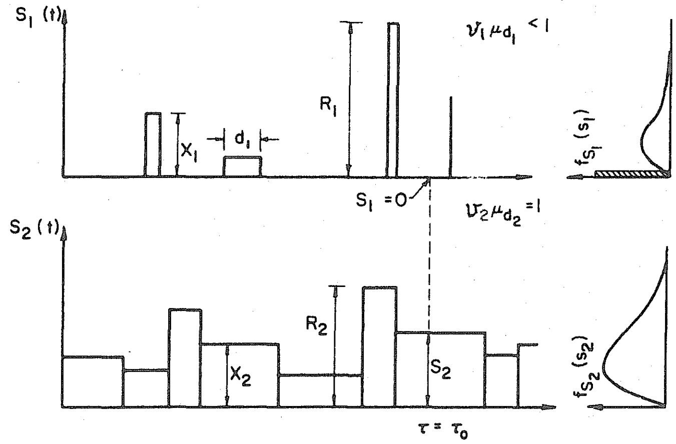

Fig. 1 Poisson Pulse Process

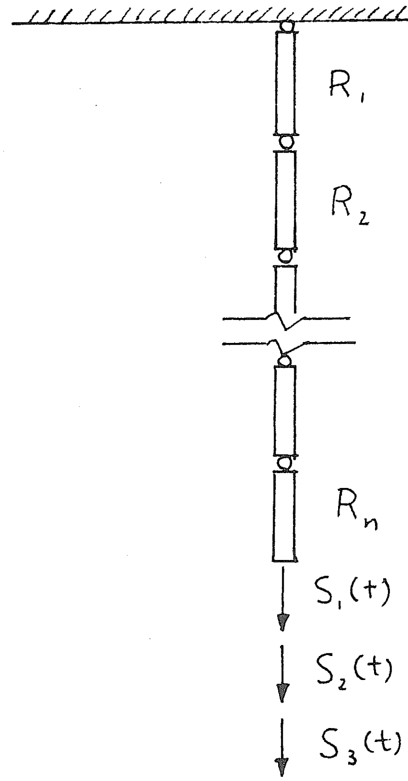

Fig. 2 Weakest Link System

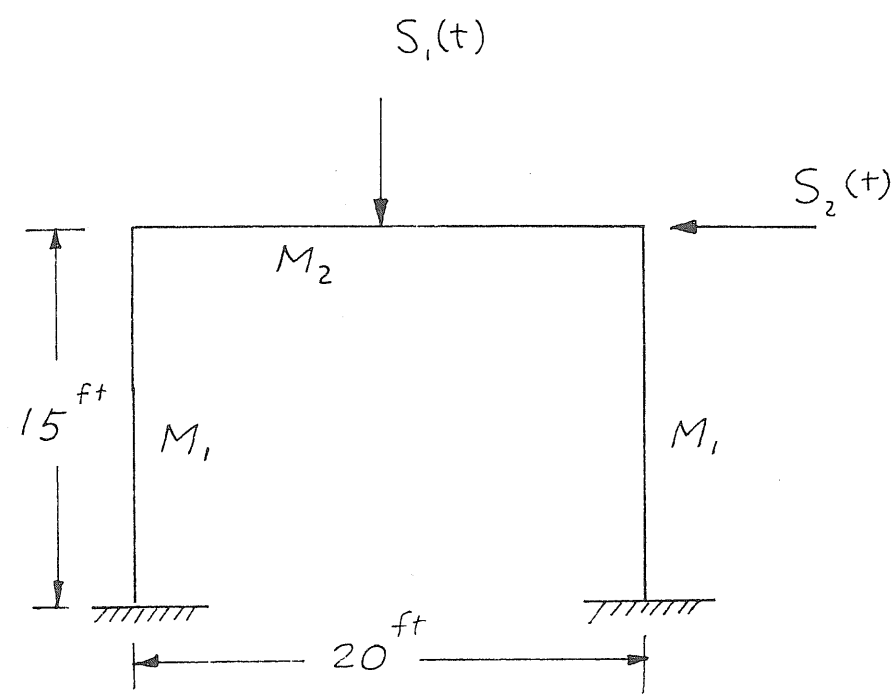

Fig. 3 Simple Ductile Frame

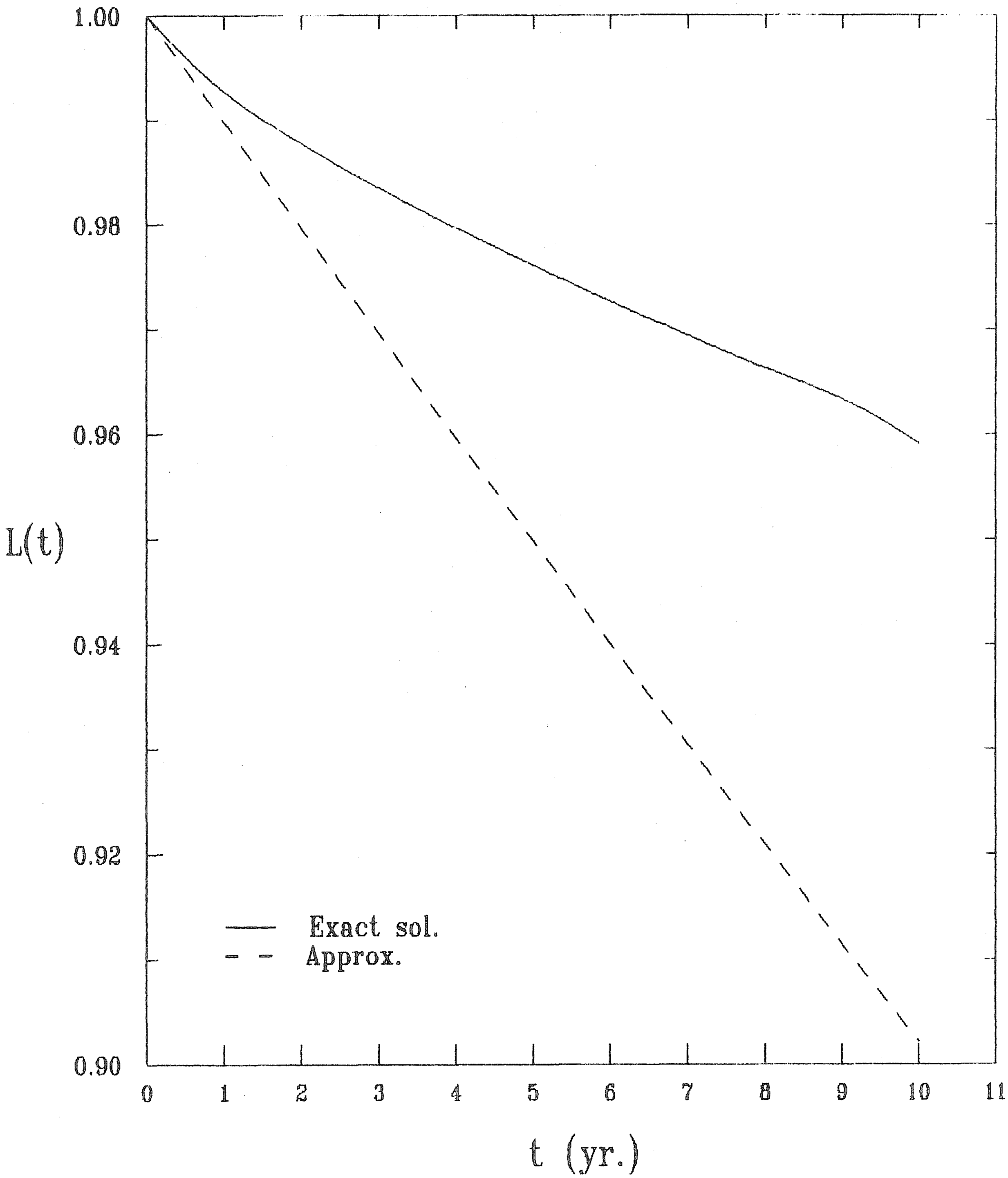

Fig. 4 Error in Using Ensemble Mean Outcrossing Rate

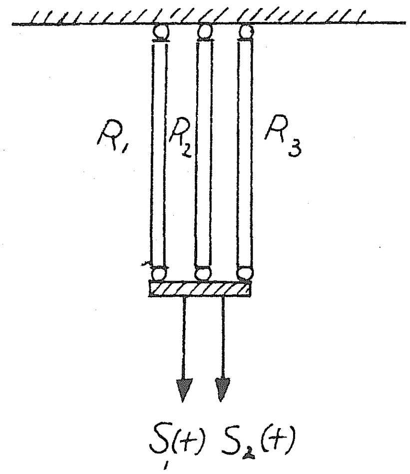

Fig. 5 System of Parallel Bars

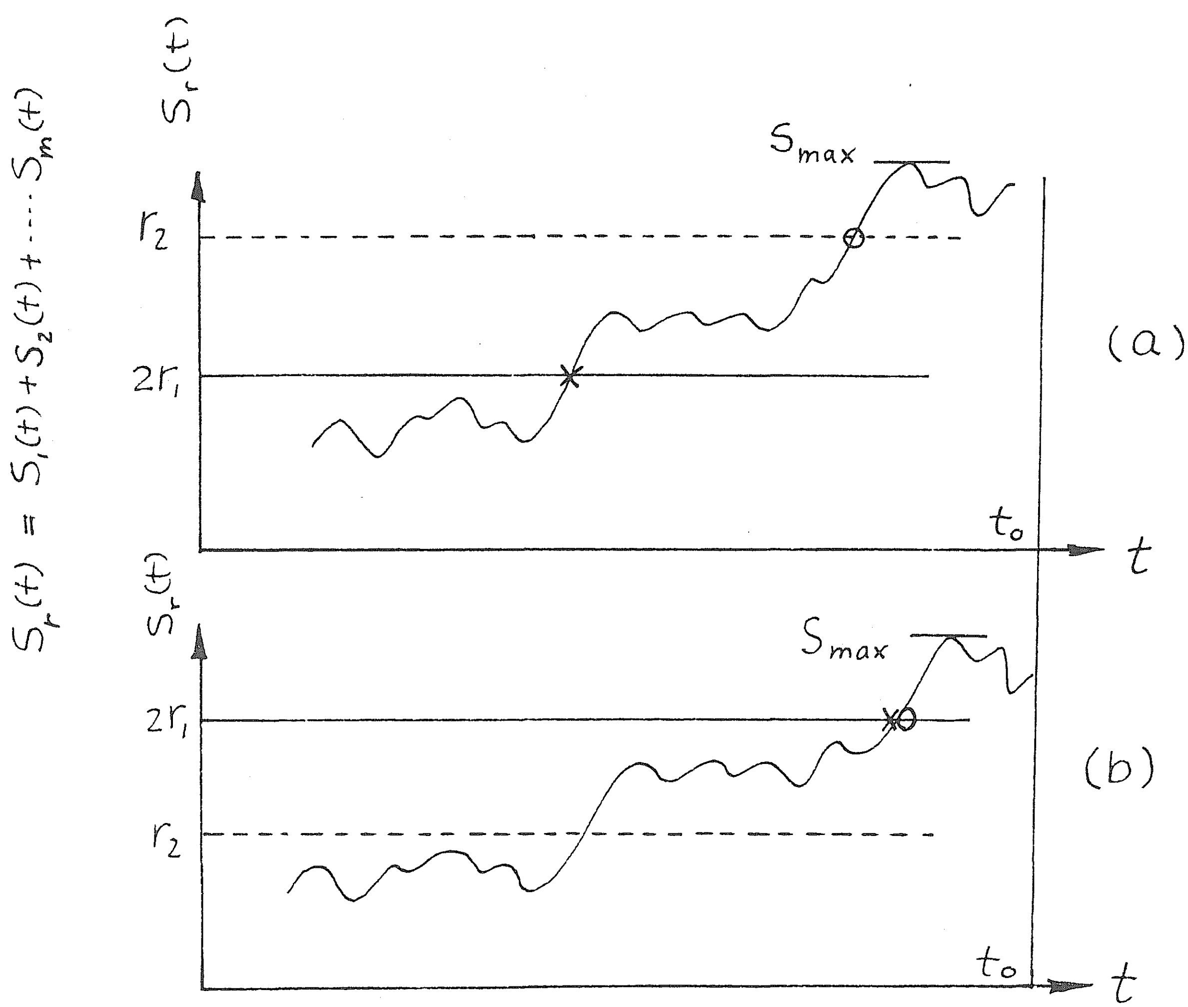

Fig. 6 Sample of Time Histories of Damage and Collapse of a Two-Bar System

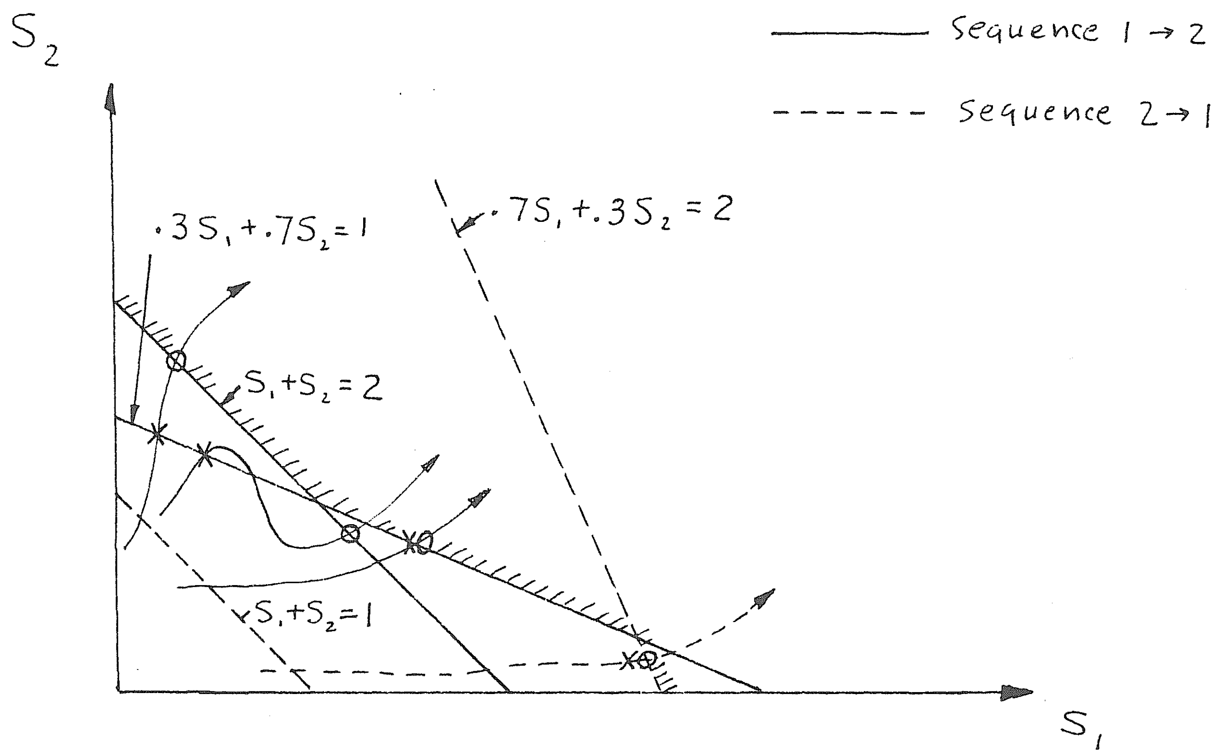

Fig. 7 Dependence of Failure Domain on Member Failure Sequence and Load Path

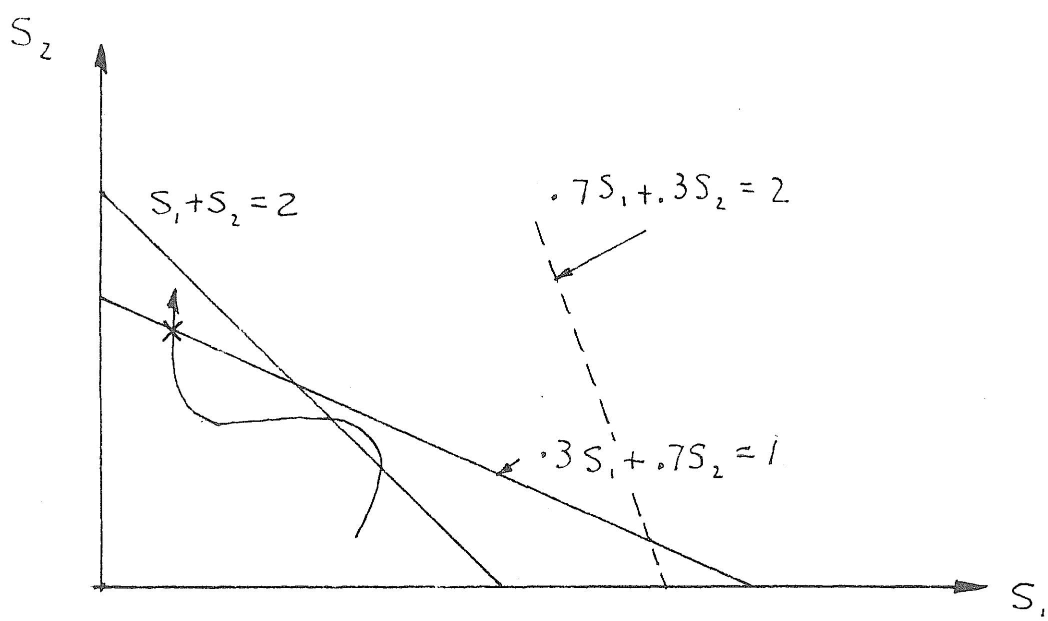

Fig. 8 Dependence of System Reliability on Load Path

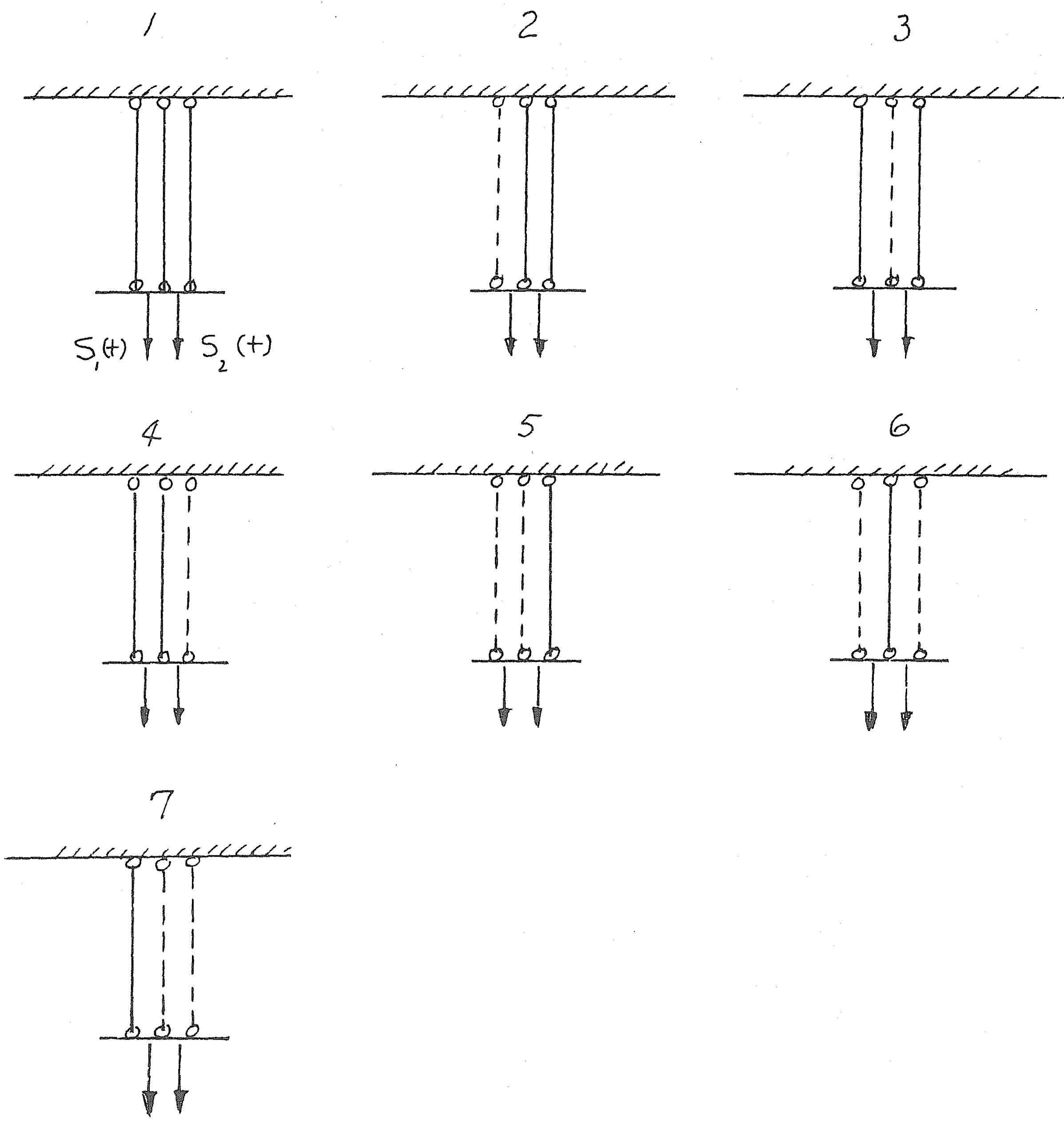

Fig. 9 Stable Configurations or Nonfailure State of a Three-Bar System

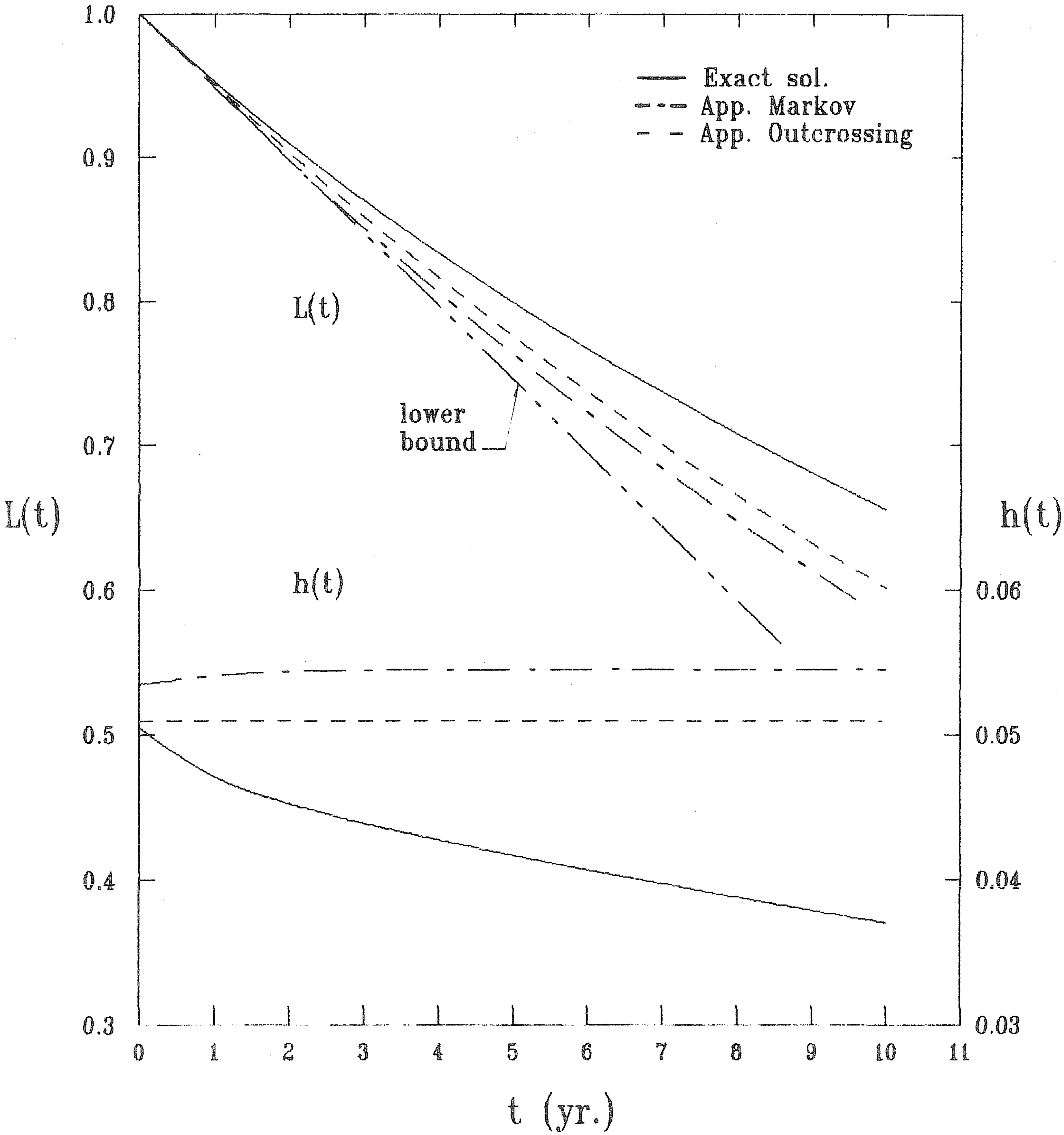

Fig. 10 Comparison of Reliability and Hazard Functions of System with Uniform Load Distribution; Resistance Uncertainty Included

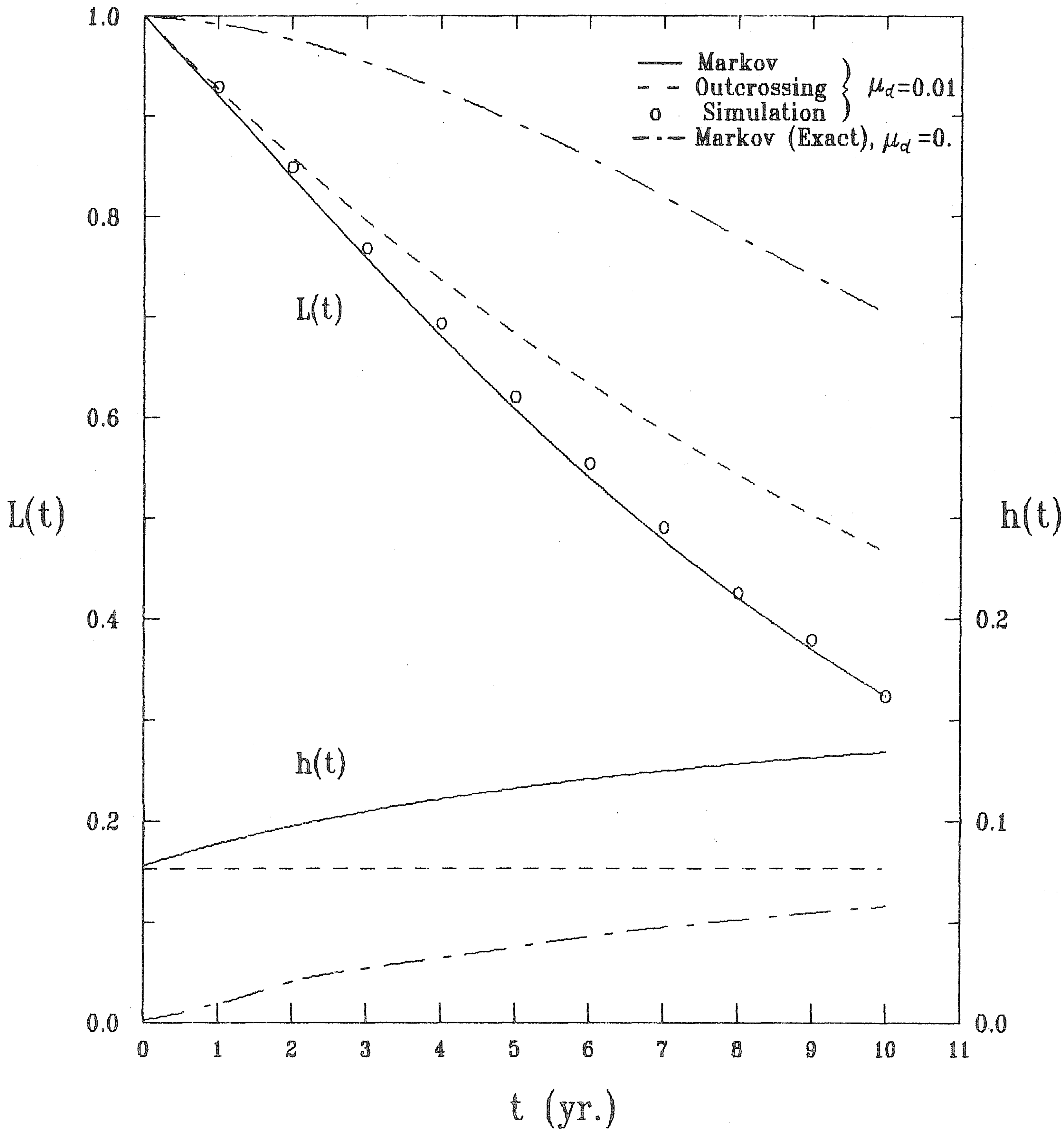

Fig. 11 Comparison of Reliability and Hazard Function of System with Nonuniform Load Distribution; Deterministic Resistance

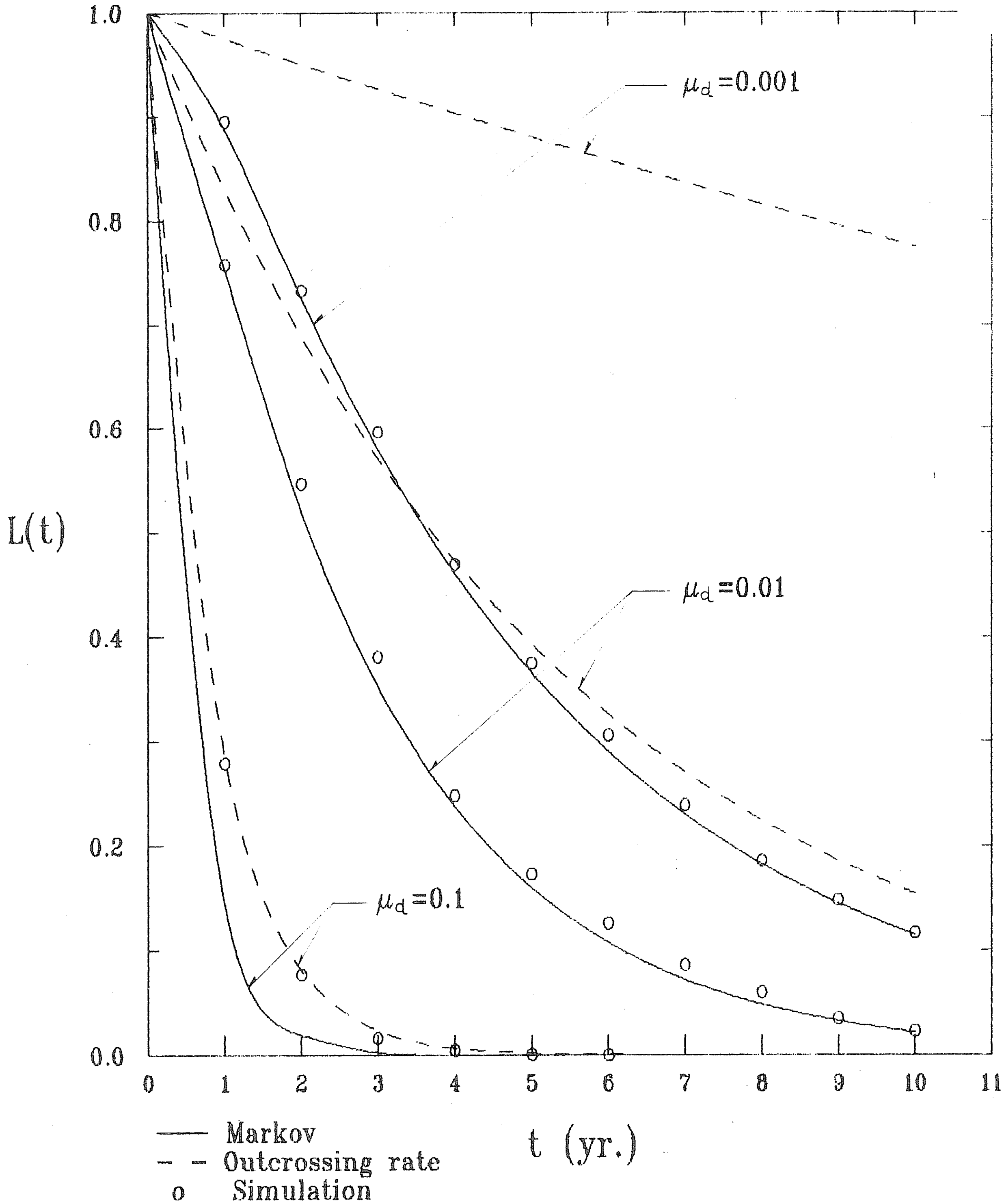

Fig. 12 Comparison of Analytical Methods with Monte Carlo Simulations (System Parameters Table 8)

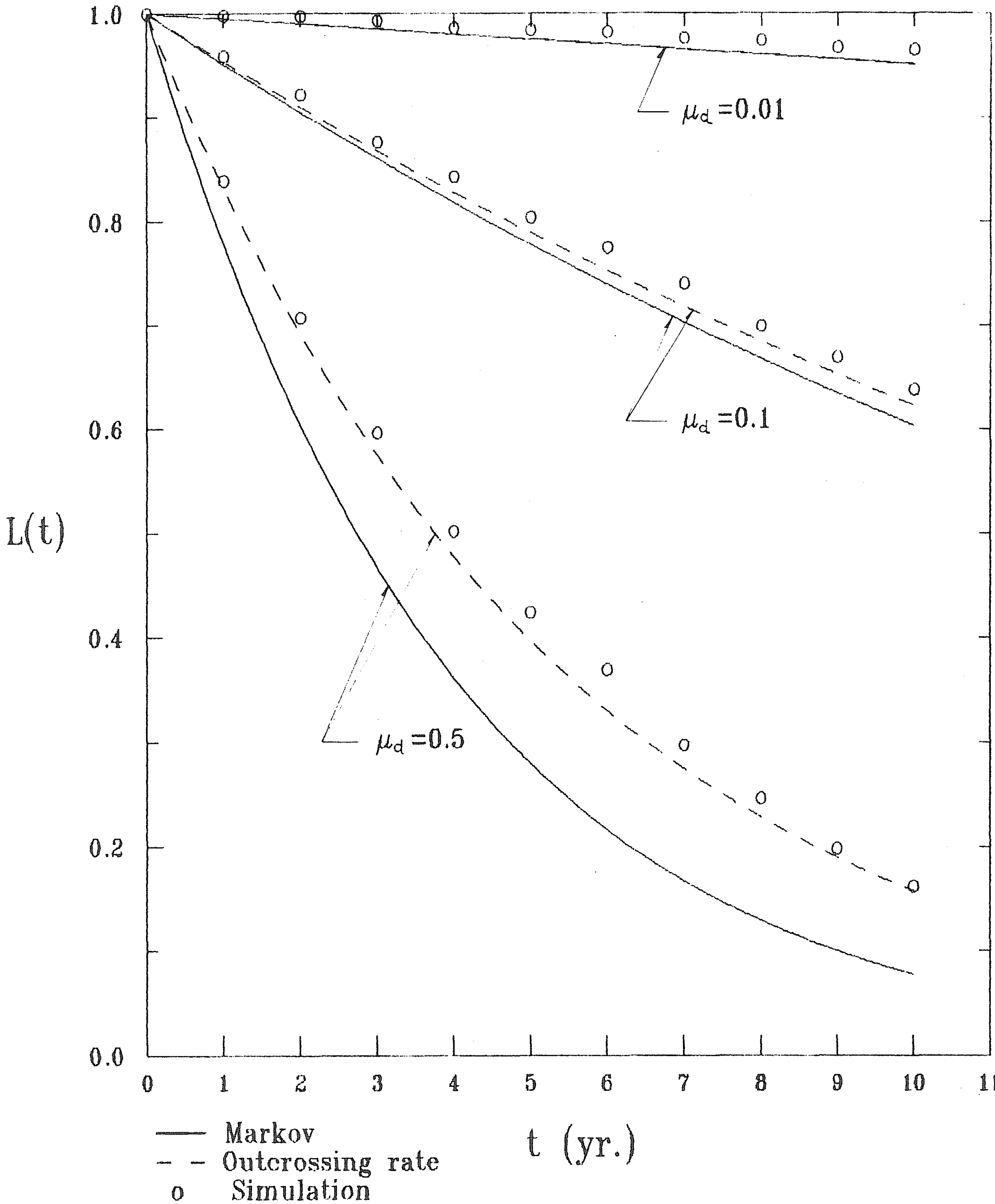

Fig. 13 Comparison of Analytical Methods with Monte Carlo Simulations (System Parameters Table 9)

APPENDIX--OUTCROSSING RATE ANALYSIS FOR SYSTEMS UNDER PULSE LOADINGS

The outcrossing rate under square wave process is according to Ref. 10.

$$\nu = \sum_{i=1}^{n} \nu_i [P(\underset{\sim}{x}^i \ \varepsilon \ F) - P(\{\underset{\sim}{x}^i \ \varepsilon \ F\} \cap \{\underset{\sim}{x} \ \varepsilon \ F\})] \qquad (A.1)$$

The pulse process can be treated as a generalization of the square wave process with a mixed density function for the load intensity, i.e., there is a probability $(1 - q_i)$, in which $q_i = \nu_i \mu_{d_i}$ that the load intensity is zero. The density of the load intensity is therefore

$$f_S(s) = (1 - \nu\mu_d)\ \delta(s) + \nu\mu_d\ f_X(s) \qquad (A.2)$$

in which $\delta(s)$ = dirac delta function and $f_X(x)$ the intensity density given that the load is on.

For the case of two load intensity and two resistance variables, using Eq. A.2 for independent load intensity

$$P(\underset{\sim}{x}_+^i \ \varepsilon \ F) = \iiiint_F [(1 - q_1)\ \delta(s_1) + q_1 f_{X_1}(s_1)]$$

$$\cdot\ [(1 - q_2)\ \delta(s_2) + q_2 f_{X_2}(s_2)]\ f_{M_1 M_2}(m_1, m_2)\ ds_1 ds_2 dm_1 dm_2 \qquad (A.3)$$

If the system consists of members in series the integration is over the failure domain

$$F = \bigcup_{i=1}^{\ell} F_i \qquad \text{and} \qquad q_i = \nu_i \mu_{d_i}$$

Expansion of Eq. A.3 gives

$$\begin{aligned} P(\underset{\sim}{x}_+^i \; \varepsilon \; F) = & (1 - q_1)(1 - q_2) \iint_F f_{M_1M_2}(m_1m_2)\, dm_1dm_2 \\ & + (1 - q_1)\, q_2 \iiint_F f_{X_2}(s_2)\, f_{M_1M_2}(m_1,m_2)\, ds_2dm_1dm_2 \\ & + q_1(1 - q_2) \iiint_F f_{X_1}(s_1) f_{M_1M_2}(m_1,m_2)\, ds_1dm_1dm_2 \\ & + q_1q_2 \iiiint_F f_{X_1}(s_1)\, f_{X_2}(s_2)\, f_{M_1M_2}(m_1,m_2)\, ds_1ds_2dm_1dm_2 \end{aligned} \qquad (A.4)$$

The multiple integral of normal density over the failure domain F can be evaluated according to the algorithm given in Ref. 4 and expressed as

$$\begin{aligned} P(\underset{\sim}{x}_+^i \; \varepsilon \; F) = & (1 - q_1)(1 - q_2)\, \Phi_3(\underline{c}_1;\underset{\approx}{\rho}_1) + (1 - q_1)\, q_2\Phi_3(\underline{c}_2;\underset{\approx}{\rho}_2) \\ & + q_1(1 - q_2)\, \Phi_3(\underline{c}_3;\underset{\approx}{\rho}_3) + q_1q_2\Phi_3(\underline{c}_4;\underset{\approx}{\rho}_4) \end{aligned} \qquad (A.5)$$

in which Φ_3 = tri-variate standard normal distribution. Similarly, one can show that

$$P(\underset{\sim}{x}_+^i \; \varepsilon \; F) - P\{(\underset{\sim}{x}_+^i \; \varepsilon \; F) \cap (\underset{\sim}{x} \; \varepsilon \; F]\} = P(\underset{\sim}{x} \; \varepsilon \; \overline{F}) - P\{(\underset{\sim}{x}_+ \; \varepsilon \; \overline{F}) \cap (\underset{\sim}{x}_+^i \varepsilon \; \overline{F})\}$$

where $\overline{F}$ = complement of F and

$$P\{(\underset{\sim}{x}_{+}^{i} \in \overline{F}) \cap (\underset{\sim}{x} \in \overline{F})\} = (1 - q_1)(1 - q_2)(1 - q_i)\Phi_6(\underline{d}_1;\underline{\underline{R}}_1)$$

$$+ (1 - q_1)q_2(1 - q_i)\Phi_6(\underline{d}_2;\underline{\underline{R}}_2) + q_1(1 - q_2)(1 - q_i)\Phi_6(\underline{d}_3;\underline{\underline{R}}_3)$$

$$+ q_1 q_2(1 - q_i)\Phi_6(\underline{d}_4;\underline{\underline{R}}_4) + (1 - q_1)(1 - q_2)q_i\Phi_6(\underline{d}_5;\underline{\underline{R}}_5)$$

$$+ (1 - q_1)q_2 q_i\Phi_6(\underline{d}_6;\underline{\underline{R}}_6) + q_1(1 - q_2)q_i\Phi_6(\underline{d}_7;\underline{\underline{R}}_7)$$

$$+ q_1 q_2 q_i\Phi_6(\underline{d}_8;\underline{\underline{R}}_8) \tag{A.6}$$

It is clear that for systems under m loads and for which n modes are considered, the evaluation of the intersection requires 2^{m+1} calculations of multi-variate normal distribution of order 2n. If the intersection term is neglected as an approximation, the order drops to n and only 2^m calculations are required. Extension of Eq. 28 for system consisting of brittle members in parallel to the case when the loadings are pulse processes can be similarly derived by substituting Eq. A.2 for the density function and expand the integrations required for the evaluation of the probabilities in Eq. 28.

PART II--APPROXIMATE METHODS FOR NONLINEAR TIME-VARIANT RELIABILITY ANALYSIS

1. INTRODUCTION

Reliability of structural systems under the action of multiple randomly time varying loads can be treated as a problem of outcrossing of a vector process of a safe domain if the structural resistance does not change with time. The probability of no "failure" is therefore that of no outcrossing over a given period of time. Exact solution of this problem is generally difficult. Various approximate methods have been developed based on a mean outcrossing rate analysis. However, determination of the mean outcrossing rate is also difficult when the safe domain is nonlinear due to nonlinear behavior of the structure or limit state function being nonlinear even when the structure remains linear. The problem can be made more tractable if one replaces the nonlinear failure surface by its Taylor series expansion and retains only the first or the first and second order terms, commonly referred to as first and second order method. Generally speaking, the accuracy of the method increases with the order, though the amount of analysis and computation required also increase. The efficiencies and ease in application of these methods to problems of practical interest are examined herein. The problem studied is the reliability of a structure with direction sensitive resistance under a wind climate which is highly directional. Numerical examples are carried out and the results based on different approximate methods are compared with "exact" solution from numerical integration.

2. FIRST ORDER (LINEARIZATION) METHOD

When only the first order terms in the Taylor series expansion of the failure surface are retained, the original nonlinear surface is replaced by a hyperplane. The outcrossing rates can be evaluated without difficulty when the loadings can be modeled as a continuous vector Gaussian process or a vector Poisson pulse process with Gaussian intensity (5,2). However, it is not immediately obvious at what point one should linearize the failure surface. The point closest to the origin in the transformed domain, where the load intensity random variables have been reduced to standard normal variates, has been suggested as the proper point (1) from an argument that it is the asymptotically correct point as the failure surface recedes from the origin. Other points, such as the point of maximum local outcrossing rate, have also been used (2). It has been shown (4), however, that the aforementioned points generally do not produce the best result because of the inherent lack of rotational symmetry in the time variant reliability problem, even in the transformed space of standard normal variates. This is due to the generally different time scales with which the component processes (individual loadings) fluctuate; e.g., their different renewal (occurrence) rates (for pulse processes) and crossing rates (for continuous process). The point which produces the best result is that which gives the stationary values of mean crossing rate out of the tangent hyperplane. For largely convex safe domain, the mean crossing rate at this point is a local maximum, whereas for concave safe domain a local minimum. This point in general does not coincide with the closest point to the origin.

Without loss of generality, for a vector renewal pulse process with standard normal intensities and renewal rate λ_i for the i-th component, the mean rate of outcrossing the hyperplane

$$\sum_{i=1}^{n} \alpha_i x_i - \beta = 0 \tag{1}$$

of this vector process is given by (2)

$$\nu = \sum_{i=1}^{n} \lambda_i \left(\phi(\beta) - \phi_2(\beta, \beta, 1 - \alpha_i^2)\right) \tag{2}$$

in which

ϕ = distribution function of a standard normal variate;

ϕ_2 = bivariate standard normal distribution;

α_i = the directional cosines of the hyperplane; and

β = the distance of the hyperplane to the origin.

For a continuous vector Gaussian process with mean function and covariance function matrices of the process and its time derivative given by

$$E\{X(t)\} = \{0\} \quad , \quad cov[X(t)] = [I]$$

$$E\{\dot{X}(t)\} = \{0\} \quad , \quad cov[\dot{X}(t)] = [\sigma_{ii}^2] \tag{3}$$

in which [I] = intensity matrix, $[\sigma_{ii}^2]$ = diagonal matrix with element σ_{ii}^2, the mean rate of outcrossing the hyperplane (Eq. 1) is (5)

$$\nu = \frac{1}{2\pi} \left(\sum_{i=1}^{n} \alpha_i^2 \, \sigma_{ii}^2 \right)^{1/2} \exp\left(- \frac{1}{2} \beta^2\right) \tag{4}$$

Therefore the best linearization point is that which gives a "stationary" value to Eqs. 2 or 4 along the failure surface. This point generally is not the closest point to the origin. The corresponding outcrossing rate is the "best" solution that linearization can give. As in the problem of finding the closest point in the time invariant problem, an algorithm may be needed to find the stationary values as the dimension of the problem increases.

3. 2ND ORDER (ASYMPTOTIC) METHODS

When higher order terms are retained in the Taylor series expansion, the approximation of the failure surface will be obviously more accurate locally. Using methods of asymptotic approximation for the evaluation of the probability integrals required, Breitung (1) proposed such an improvement by including the curvatures of the failure surface at the expansion point. In essence, it replaces the failure surface by a second order parabolic one at the point closest to the origin in the transformed space of standard normal variates. This method can be applied to both time invariant and time variant problems. For the foregoing loadings modeled by a standard vector normal process, the mean outcrossing rates according to this "parabolic" approximation are results given by Eqs. 2 and 4, respectively, multiplied by a factor

$$C = \prod_{i=1}^{n-1} \left(1 - \beta\ \kappa_i^{\beta}\right)^{-1/2} \tag{5}$$

in which κ_i = the main curvatures of the surface at the expansion point. Therefore, additional efforts are required to calculate κ_i. Breitung (1) has shown that the result is asymptotically correct, i.e., it approaches the exact solution as $\beta \to \infty$.

Generalizing the results for the extreme value of a χ^2-process, Lindgren (3) arrived at the asymptotic outcrossing rate (hence also the extreme value distribution based on a Poisson outcrossing assumption for high threshold levels) of quadratic surface of a continuous vector Gaussian process with independent components. For a failure surface given by the following quadratic equation,

$$g(x_1, x_2, \ldots x_n) = \sum_{i=1}^{P} (x_i - a_i)^2 + \sum_{i=P+1}^{n} \gamma_i (x_i - a_i)^2 - r^2 = 0 \tag{6}$$

in which $\gamma_i < 1$, as $r \to \infty$, the outcrossing rate of a standard normal process (described by Eq. 3) is given by

$$\nu \approx \mu(r) \prod_{i=P+1}^{\pi} (1 - \gamma_i)^{-1/2} \exp[\tfrac{1}{2}\, a_i^2\, \gamma_i/(1 - \gamma_i)] \tag{7}$$

in which $\mu(r)$ is the mean outcrossing rate of the vector process of the failure surface

$$\sum_{i=1}^{P} (x_i - a_i)^2 = r^2 \tag{8}$$

When the time scales in the component processes are equal, i.e., $\sigma_{\dot{x}_i} = \sigma_{\dot{x}}$ for i = 1 to n

$$\mu(r) = \frac{1}{2\pi} \sigma_{\dot{x}} \left(\frac{r}{r_o}\right)^{\frac{P-1}{2}} \exp\left[-\left(\frac{r-r_o}{2}\right)^2\right] \tag{9}$$

in which $\quad r_o = \left(\sum_{i=1}^{P} a_i^2\right)^{1/2}$

Otherwise, a numerical integration is needed to find the scale factor in Eq. 9.

The basis for Eq. 6 is that for $\gamma_i < 1$, the asymptotic outcrossing is dominated by that out of the lower dimensional sphere (Eq. 8); therefore, the asymptotic outcrossing rate is equal to that out of the sphere modified by a factor which is a function of the parameters of the rest of the dimensions but independent of the threshold level.

Lindgren's result obviously can be used in conjunction with the 2nd order approximation. If one retains all the 2nd order terms in the Taylor series expansion of a general failure surface, one has a quadratic surface. With a proper transformation, this surface can be reduced to the standard form given by Eq. 6, hence the asymptotic outcrossing rate or extreme value distribution can be evaluated. Henceforth, this is referred to as "quadratic" approximation. Comparing Eqs. 4 and 5 with

Eq. 7, one sees many similarities. The quadratic approximation is slightly more general in that it is not restricted to use any specific form of the substitute failure surface, such as a paraboloid.

4. RELIABILITY OF STRUCTURES UNDER WIND

Wind fluctuates in intensity and direction. The two horizontal components of wind velocity have been shown to be approximately Gaussian processes, therefore the wind velocity can be treated as a vector Gaussian process. The statistics of this vector process such as the mean values, covariance and spectral density matrices specify the intensity and directionality of the wind. Structural capacity against wind (resistance) is often direction-dependent because of the possible asymmetry of the structural geometry as well as possible significant fluid-structure interaction. The reliability of the structure under wind when the wind direction is included in the consideration therefore can be treated as a problem of outcrossing of the vector wind velocity process of a nonlinear failure surface dependent on the structural resistance. This problem has been investigated in Ref. 6. It is a problem of practical interest, with small dimension (n = 2) and a highly nonlinear failure surface, therefore is an appropriate test problem for the purpose of this study.

The structure is assumed to have a direction-sensitive response (6), described by

$$r = C\,U^2\left[\cos^2\theta + \frac{\sin^2\theta}{a}\right]^{-1/2} \tag{10}$$

in which

$U(t)$ = wind speed = $[X^2(t) + Y^2(t)]^{1/2}$;

$X(t), Y(t)$ = the two components of wind velocity modeled as Gaussian processes;

$\theta(t) = \alpha(t) - \beta$ = wind direction angle relative to the structure orientation;

$\alpha(t)$ = wind direction = $\tan^{-1} \frac{Y(t)}{X(t)}$; and

β = structural orientation angle.

See Fig. 1 for parameter definitions.

For a given wind speed, Eq. 10 is an ellipse with respect to θ. a controls the sensitivity of the response to change of wind direction (Fig. 2). For a given allowable response threshold $r = r_o$, the failure surface according to Eq. 10, in Cartesian coordinates is given by

$$r(X,Y) = (X^2 + Y^2)^{1/2} (C_1 X^2 + C_2 Y^2 + C_3 XY) = r_o \qquad (11)$$

in which,

$$C_1 = \cos^2\beta + \frac{1}{a} \sin^2\beta$$

$$C_2 = \sin^2\beta + \frac{1}{a} \cos^2\beta$$

$$C_3 = 2\left(1 - \frac{1}{a}\right) \sin\beta \cos\beta$$

The surface described by Eq. 11 is highly nonlinear with convex as well as concave segments, therefore it presents a relatively rigorous test of the approximate methods.

The wind climate statistics are those of Ref. 6 shown in Table 1. It represents the general condition at Airport Weather Station, Baltimore, MD. It is somewhat dominated by the components in the northwest direction; i.e., at $\alpha = \alpha_o$. The wind velocity process is first transformed into a standard form, i.e., related to the original process by

$$\{\underset{\sim}{X}\} - \{\underset{\sim}{\mu}_x\} = [R]\{\underset{\sim}{V}\} \tag{12}$$

and

$$\{\dot{\underset{\sim}{X}}\} = [R]\{\dot{\underset{\sim}{V}}\} \tag{13}$$

in which

$\{\underset{\sim}{X}\}$ = velocity vector process;

$\{\underset{\sim}{\mu}_x\}$ = mean function column matrix;

$\{\underset{\sim}{V}\}$ = standard normal vector process described by Eq. 3; and

$[R]$ = transformation matrix.

The corresponding elements in the matrix $\text{cov}[\dot{\underset{\sim}{V}}]$ and the transformation matrix R are given in Table 2. The failure surfaces for several combinations of response threshold and structural orientation angle in the transformed space are shown in Figs. 3, 4 and 5 for the case a = 0.1.

5. NUMERICAL RESULTS AND COMPARISON

In both the first order and the 2nd order methods, partial derivatives of the failure surface function are needed in determining the approximate surfaces or the main curvatures. In the Appendix, the derivatives are shown to give an indication of the analytical and numerical effort required for each method. To locate the expansion point, as mentioned in the foregoing, an algorithm is needed in both the linearization method and the "parabolic" approximation. Similar problems exist in the "quadratic" approximation, i.e., at what point should one do the expansion. Several points are tried, i.e., at the point of mean values, the point of extreme values (such as mean plus three standard deviations) as well as the point closest to the origin. It is found that depending on the values of the parameters of the problem as well as the choice of the expansion point, the resultant quadratic surface can be elliptic or hyperbolic and the outcrossing rate could be widely different and quite sensitive to such choices. As in the time invariant problem, the outcrossing depends on the probability content outside the approximate safe domain, which is highest when the point is closest to the origin. For this reason and also for easy comparison with the "parabolic" approximation, the closest point is used. Furthermore, a linear transformation is needed to reduce the general quadratic equation into the standard form given in Eq. 6. Some details of the analysis are given in the Appendix.

For purposes of comparison, the outcrossing rate is also evaluated numerically using the outcrossing rate formula from Ref. 6 based on an extension of Rice's result.

$$\nu = \nu_o \sqrt{2\pi}\, \sigma_U \int_0^{2\pi} f_{U,\alpha}\left[g,\theta+\beta\right] \sqrt{1 + \left(\frac{dg}{d\theta}/g\right)^2}\, d\theta \tag{14}$$

in which ν_o = cyclic rate of wind speed, $f_{U,\alpha}$ = joint density function, and g = wind speed threshold function (inverse function of Eq. 10).

The case with a = 0.1 and three different structural orientation angles is studied. $\beta = \alpha_o$ corresponds to the most unfavorable orientation, i.e., aligning the weak axis of the structure with the dominant wind direction. The outcrossing rates based on the approximate methods are compared with the "exact" (numerical) solution in Table 3. In this particular problem, the best linearization point is very near the closest point. The corresponding approximate failure surfaces are shown in Figs. 3, 4 and 5. The quadratic function at the closest point is a hyperbola. The corresponding rates of outcrossing the approximate (linear and parabolic) failure surfaces at the second expansion points are also shown in brackets. It is seen that the two second order asymptotic solutions are very close, both consistently underestimate the outcrossing rate due to the nature of the approximate failure surfaces. The linearization on the other hand, may give over or underestimate, and in general gives better results. These somewhat unexpected results can be attributed to the particular shape of the failure surface of this problem. The outcrossing rates out of the approximate failure surface at the second expansion points are found to be generally much smaller. The outcrossing rates at these two expansion points may be added to give a conservative estimate of the vector process outcrossing the region enclosed by the two approximate failure surfaces. The distributions of the annual maximum response based on this combined outcrossing rate and a Poisson outcrossing assumption

$$P(R_{max} < r) \approx \exp[-\nu(r)t]$$

are obtained and compared in Fig. 6 with the results based on the exact outcrossing rates.

6. SUMMARY AND CONCLUSIONS

Approximate methods for nonlinear time variant reliability analysis are examined. They are, in increasing order of level of analysis and computation required, first order (linearization) method, second order asymptotic methods with parabolic and quadratic approximation, respectively. Emphasis is on accuracy and ease in application to problems of practical engineering interest. Numerical results of reliability of a structure with direction sensitive resistance under vector wind force are obtained and compared. It is found that these approximate methods, if intelligently used, give satisfactory results. The two second order asymptotic methods are very similar in nature and produce results of comparable accuracy. The quadratic approximation, however, is more general and requires more analytical and computational effort. Also, although higher order methods give better approximation of the failure surface locally, they might not always give more accurate results depending on the characteristics of the failure surface as demonstrated by the numerical examples given herein. One still needs to be vigilant when using these second order methods.

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TABLES

Table 1

Summary of Statistics of One-Minute Wind at Baltimore WBAS

N-S Component (Y) mph		E-W Component (X) mph		Correlations Coefficients	Dominant Direction	Cyclic Rate (per day)
μ_y	σ_y	μ_x	σ_x	$\rho_{x,y}$	α_o	ν_o
-.69	6.55	3.20	7.75	-0.143	159°	2.47

$$\mathrm{cov}[\dot{X}(t)] = \begin{bmatrix} \sigma^2_{\dot{x}\dot{x}} = 25.37 & \sigma^2_{\dot{x}\dot{y}} = -5.33 \\ \sigma^2_{\dot{x}\dot{y}} = -5.33 & \sigma^2_{\dot{y}\dot{y}} = 40.44 \end{bmatrix}$$

Table 2

Transformation Matrix and Covariance Matrix of $\dot{V}$

$$[R] = \begin{bmatrix} -7.721 & .375 \\ -.666 & 6.539 \end{bmatrix} \qquad \mathrm{cov}[\dot{\underset{\sim}{V}}(t)] = \begin{bmatrix} .418 & 0 \\ 0 & .944 \end{bmatrix}$$

Table 3

Mean Outcrossing Rate

(Outcrossings/hr) $\nu \times 10^4$

Normalized Threshold Level	Linearization	Parabolic Approx.	Quadratic Approx.	Exact Sol.
		$\beta = \alpha_o$		
1.0	1.809 (.068)	1.046 (.037)	1.004	1.374
1.2	.429 (.011)	.247 (.006)	.238	.328
1.4	.100 (.002)	.057 (.001)	.056	.083
		$\beta = \alpha_o + \pi/4$		
.80	1.333 (.092)	.711 (.043)	.674	1.495
.90	.529 (.031)	.281 (.014)	.268	.492
1.20	.032 (.001)	.017 (.0005)	.016	.035
		$\beta = \alpha_o + \pi/2$		
.65	.594 (.308)	.239 (.122)	.232	.99
.70	.322 (.164)	.129 (.064)	.127	.375
.90	.028 (.013)	.011 (.005)	.011	.011

FIGURES

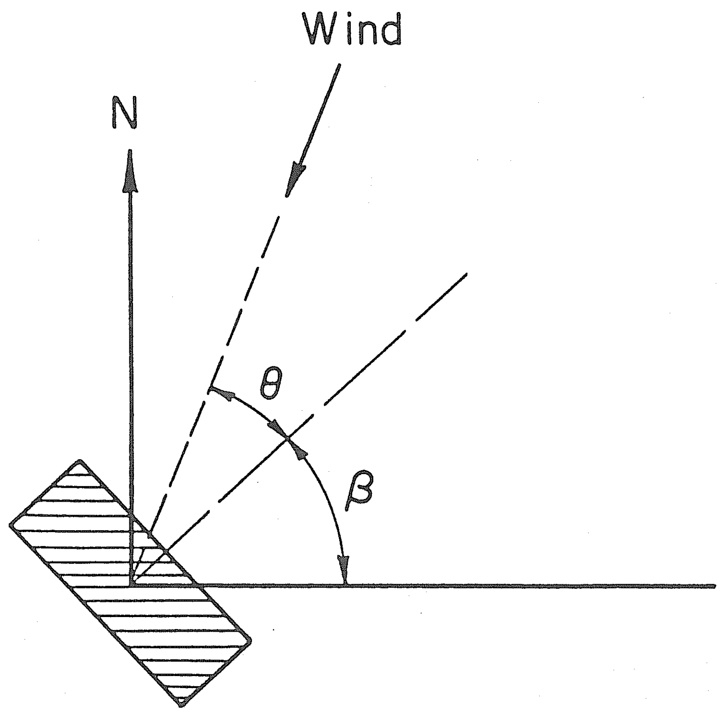

Fig. 1 Problem Geometry

$$X(\theta) = (\cos^2\theta + \sin^2\theta / a)^{-1/2}$$

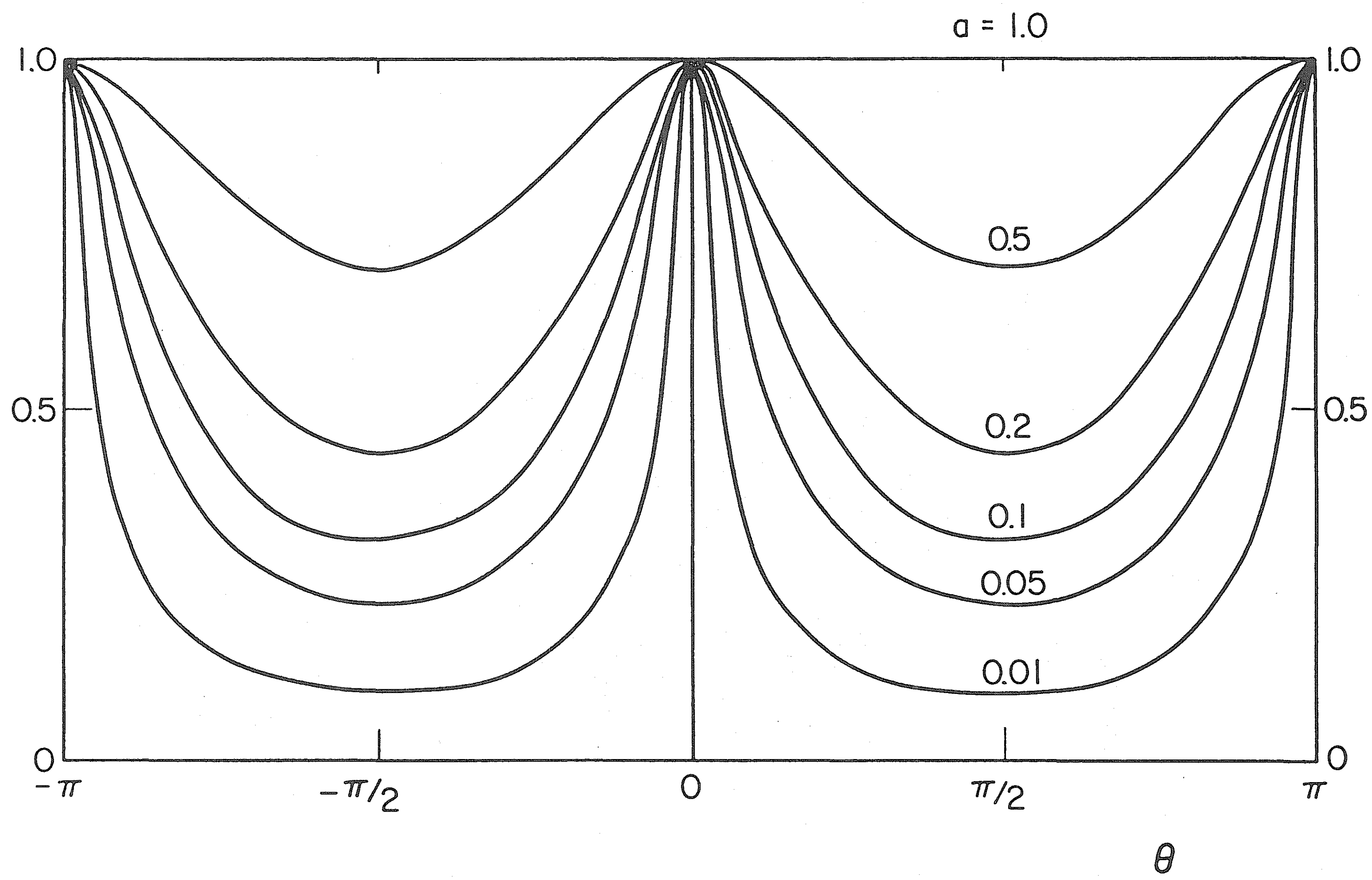

Fig. 2 Response as Function of Wind Direction

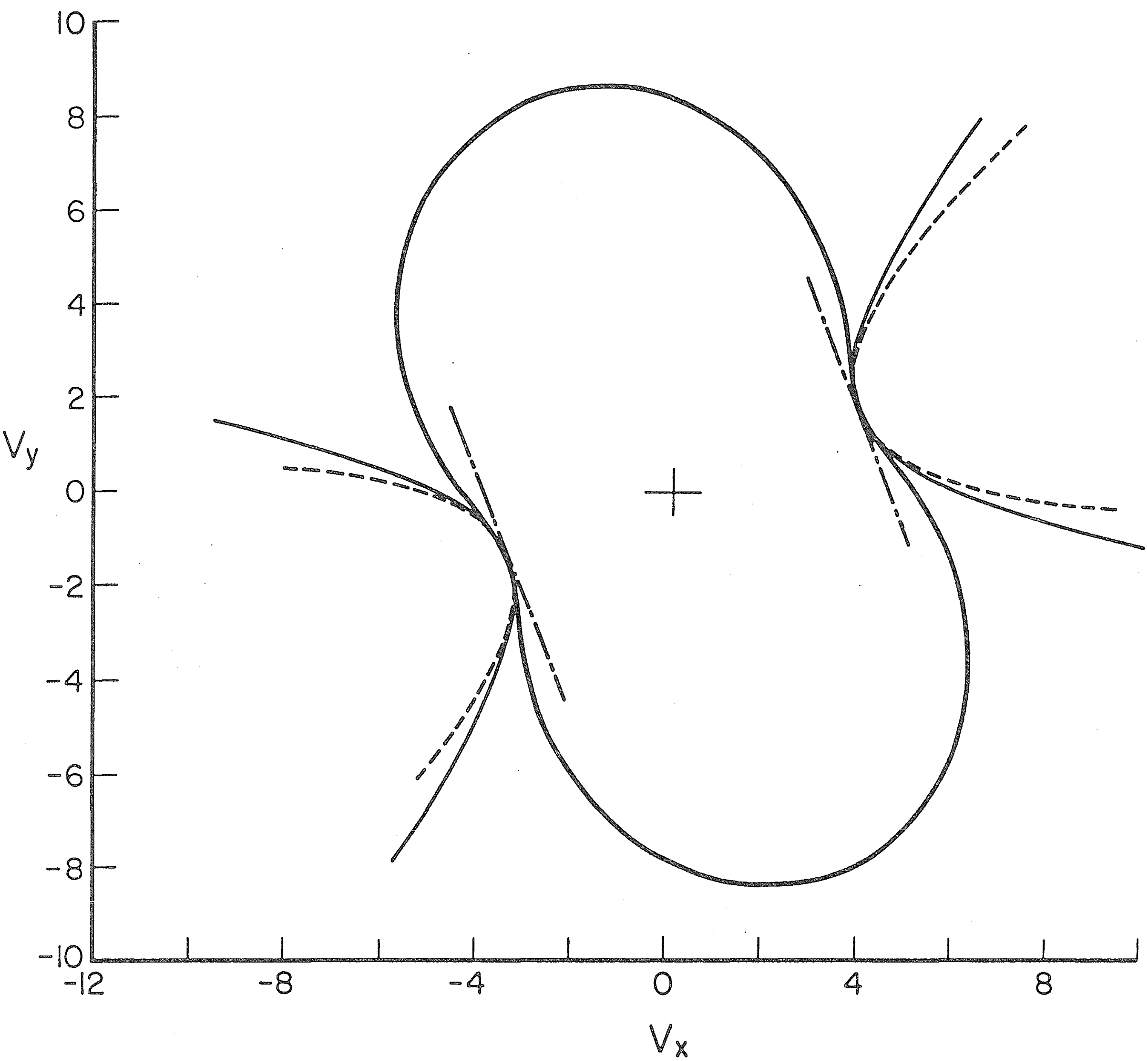

Fig. 3 Failure Surface and its Approximations in the Transformed Space of Standard Normal Variates

-·-·-· First Order (Linearization) Approximation

------ Parabolic Approximation

——— Quadratic Approximation

($\beta = \alpha_o$, $r = 1.0$)

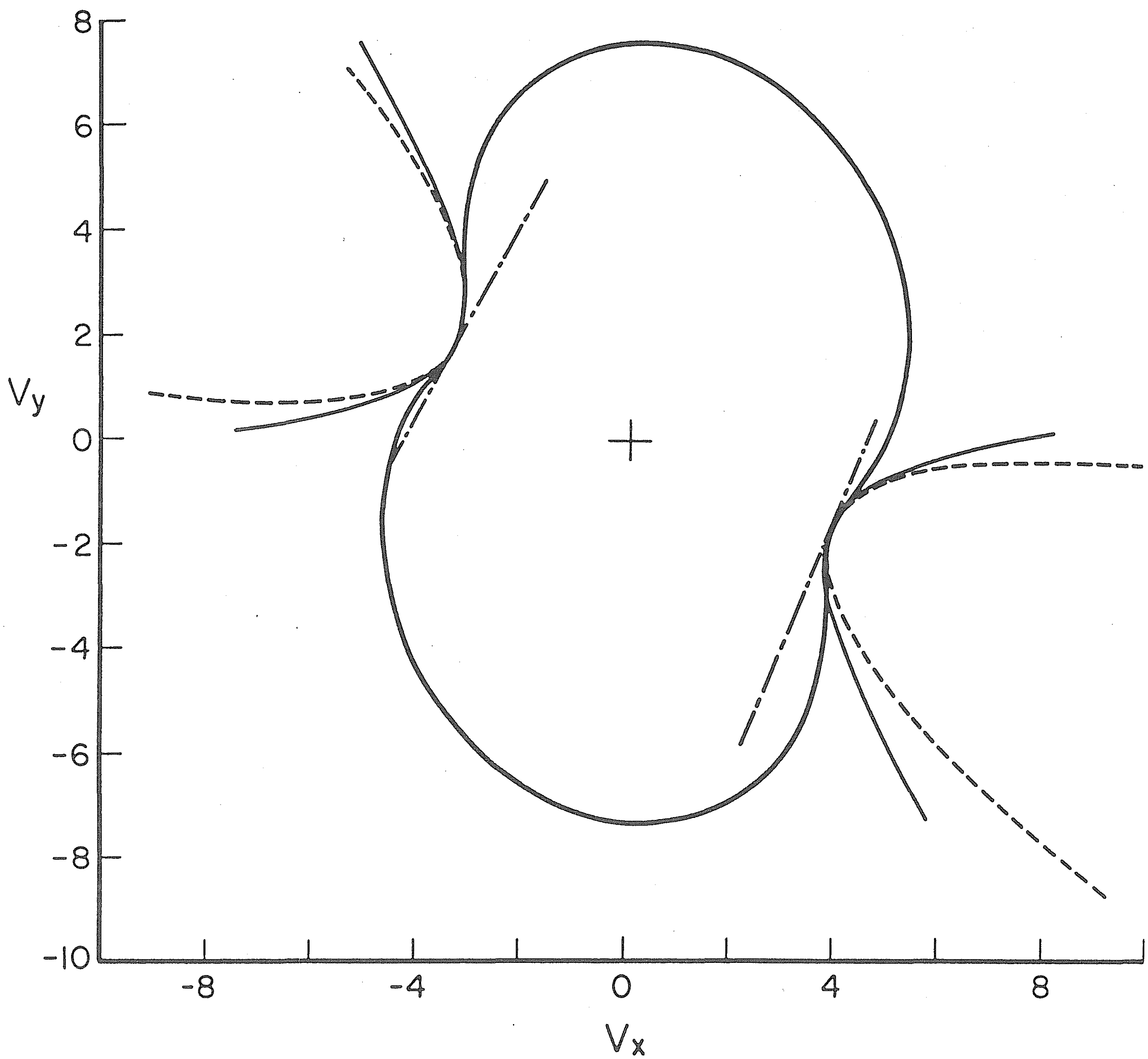

Fig. 4 Failure Surface and its Approximations in the Transformed Space of Standard Normal Variates

-·-·-· First Order (Linearization) Approximation

------ Parabolic Approximation

——— Quadratic Approximation

($\beta = \alpha_o + \pi/4$, $r = 0.8$)

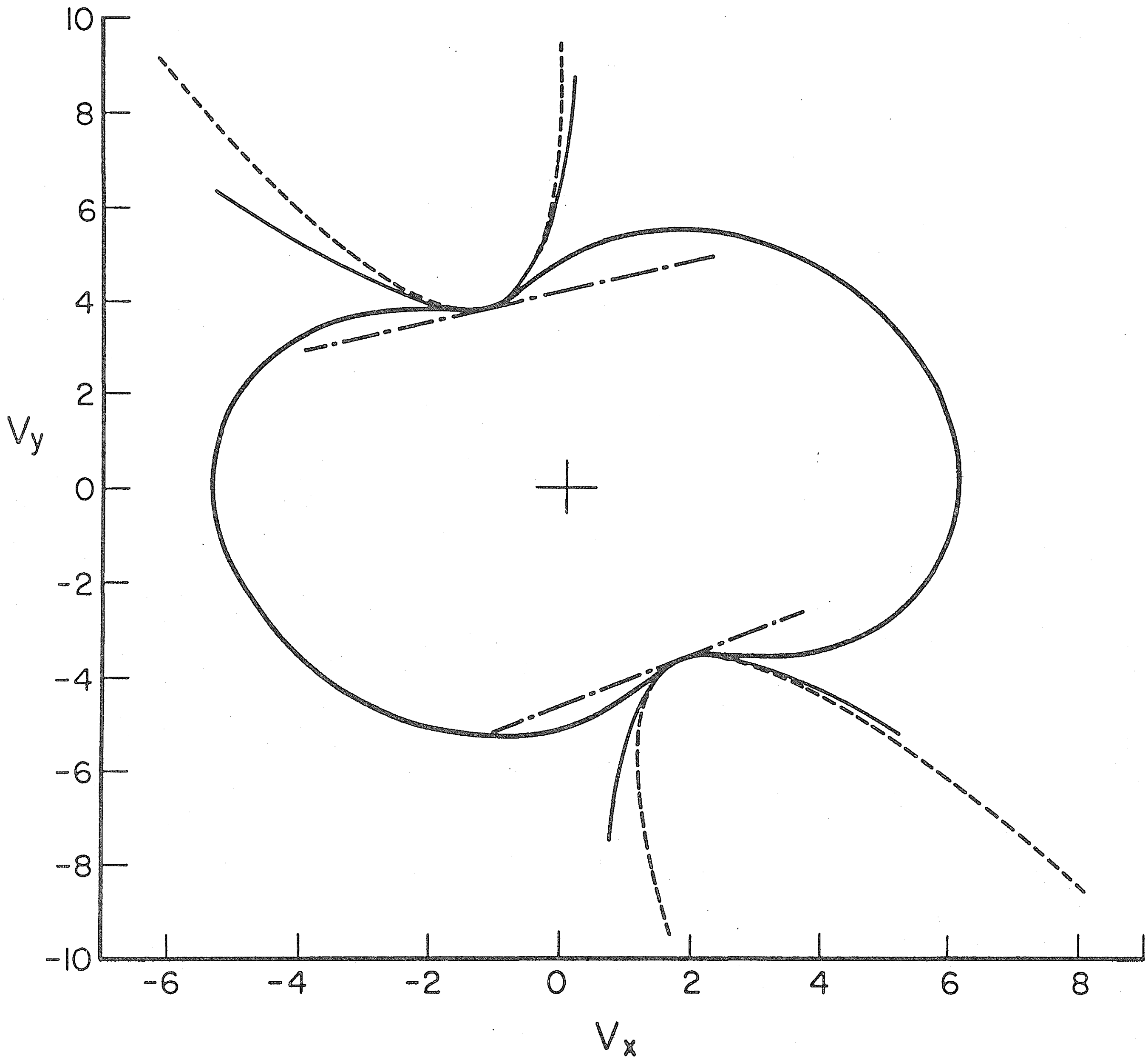

Fig. 5 Failure Surface and its Approximations in the Transformed Space of Standard Normal Variates

-·-·-· First Order (Linearization) Approximation

------ Parabolic Approximation

——— Quadratic Approximation

$(\beta = \alpha_o + \pi/2 \quad , \quad r = 0.65)$

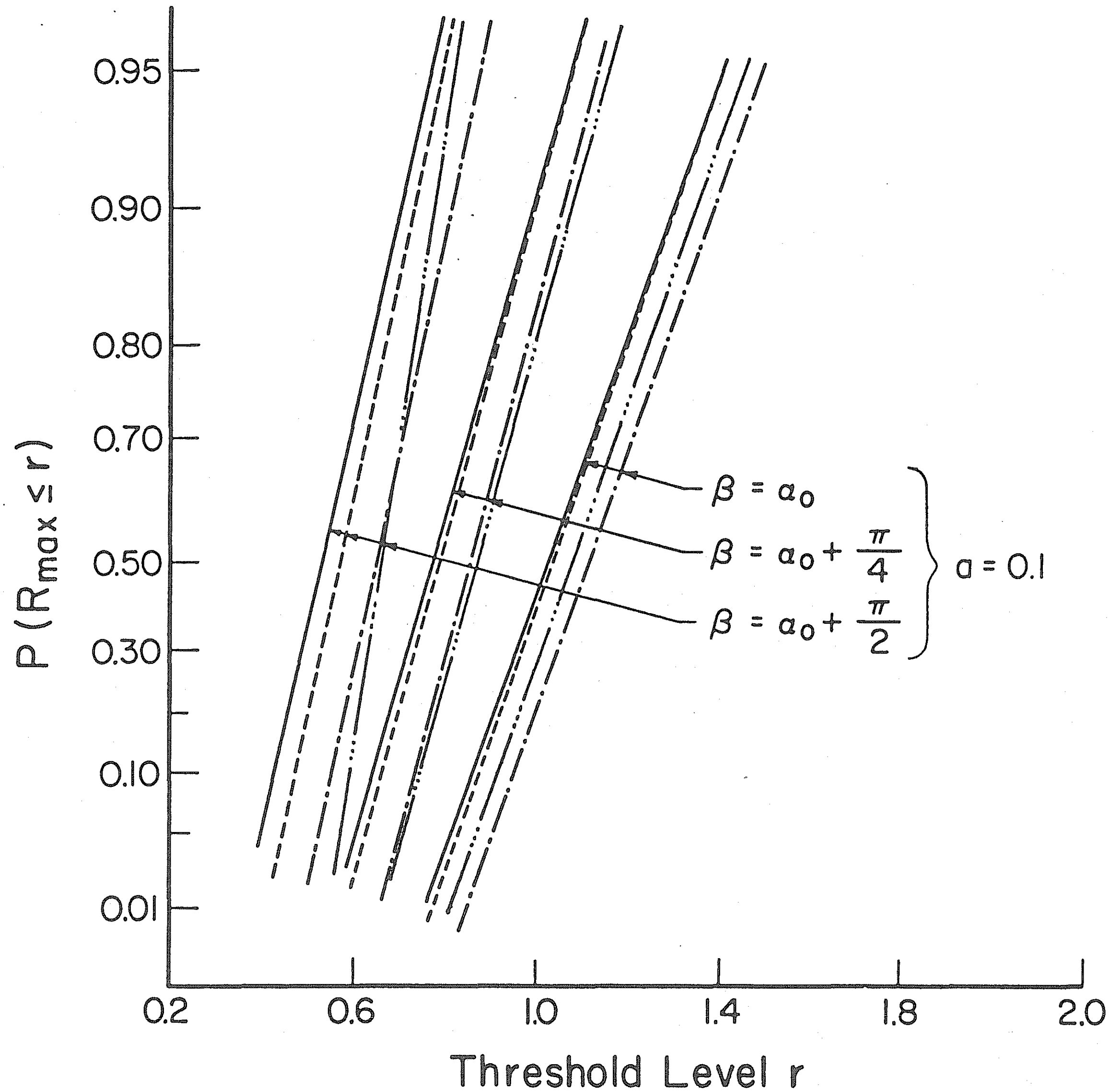

Fig. 6 Distribution of Annual Maximum Response

—···—···— Exact Solution

—·—·—·—· First Order Approximation

------------ Parabolic Approximation

———————— Quadratic Approximation

APPENDIX--DETAILS OF DERIVATION

The function value and partial derivatives of Eq. 11 at $x = x_o$ and $y = y_o$ required for the first and second order approximations are:

$$r_o = E^{3/2} F^{-1/2}$$

$$r_x = E^{1/2} F^{-3/2} \left[3x_o F - \frac{1}{2} EG\right]$$

$$r_y = E^{1/2} F^{-3/2} \left[3y_o F - \frac{1}{2} EH\right]$$

$$r_{xx} = 3E^{-1/2} F^{-3/2} \left[F(x_o^2 + E) - x_o EG\right] + E^{3/2} F^{-5/2} \left(\frac{3}{4} G^2 - C_1 F\right)$$

$$r_{yy} = 3E^{-1/2} F^{-3/2} \left[F(y_o^2 + E) - y_o EH\right] + E^{3/2} F^{-5/2} \left(\frac{3}{4} H^2 - C_2 F\right)$$

$$r_{xy} = 3E^{-1/2} F^{-3/2} \left[x_o y_o F - \frac{1}{2} x_o EH - \frac{1}{2} y_o EG\right] + \frac{1}{2} E^{3/2} F^{-5/2}$$

$$+ \left[\frac{3}{2} GH - C_3 F\right]$$

in which

$$E = x_o^2 + y_o^2 \quad , \quad F = C_1 x_o^2 + C_2 y_o^2 + C_3 x_o y_o$$

$$G = 2 C_1 x_o + C_3 y_o \quad , \quad H = 2 C_2 y_o + C_3 x_o$$

The 2nd order expansion of the function is

$$r(x,y) \approx r_o + r_x(x - x_o) + r_y(y - y_o) + \frac{1}{2}\,[r_{xx}(x - x_o)^2$$

$$+ r_{yy}(y - y_o)^2 + 2\, r_{xy}(x - x_o)(y - y_o)]$$

Through an orthogonal transformation

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = [T] \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

the quadratic equation is reduced to the standard form that Lindgren's asymptotic solution applies

$$r(u_1, u_2) = r_o + b_1 u_1 + b_2 u_2 + b_3 u_1^2 + b_4 u_2^2 = a$$

For example for $b_3 > b_4$, Lindgren's asymptotic solution gives

$$P\{\mathrm{Max}[(u_1 + \frac{b_1}{2b_3})^2 + \frac{b_4}{b_3}\,(u_2 + \frac{b_2}{2b_4})^2]$$

$$\leq \frac{a}{b_3} - \frac{r_o}{b_3} + \frac{b_1^2}{4b_3^2} + \frac{b_2^2}{4b_4 b_3} = u_o^2 \text{ in } (0,T)\}$$

$$= \exp\{-\,(1 - \frac{b_4}{b_3})^{-1/2}\,\frac{T\sqrt{\lambda}}{2\pi}\,\exp[-\,\frac{1}{2}\,(u_o + \frac{b_1}{2b_3})^2$$

$$+ \frac{1}{2}\,(\frac{b_2}{2b_4})^2\,\frac{b_4}{b_3}/(1 - \frac{b_4}{b_3})]\}$$

in which $\lambda = \sigma^2_{\dot{u}_1}$

Switch b_4 and b_3 if $b_3 < b_4$. Note that it is of the double exponential form.